# Conservation laws and nuclear reactions at low energies

- Charge
- (energy, momentum) four momentum
- Angular momentum

  (*orbital angular momentum and spin*)
- Parity
- Number of nucleons

  *only at low energies: E < $mc^2$*

are related, orbital angular momentum of nucleus wave function determines parity

$$P = (-1)^L$$

- Baryon number
- Lepton number

# Energy conservation, Q value

Relativistic conservation of energy

$$m_p c^2 + T_p + m_T c^2 + T_T = m_R c^2 + T_R + m_e c^2 + T_e$$

Q value:

$$Q = \left(m_p + m_T - m_R - m_e\right)c^2 = \left(m_{initial} - m_{final}\right)c^2$$

$$= T_R + T_e - T_T - T_p = T_{final} - T_{initial}$$

# Energy conservation, Q value

with $c = 1$

Q value: $Q = m_p + m_T - m_R - m_e$

Energy is conserved

$$Q = T_R + T_e - T_T - T_p$$

$T$ = kinetic energy

Standard, fixed target kinematics: target at rest

Lab system $T_T = 0$

$Q > 0$ exothermic reaction

$Q < 0$ endothermic reaction

$Q = 0$ elastic scattering $a + A \rightarrow a + A$

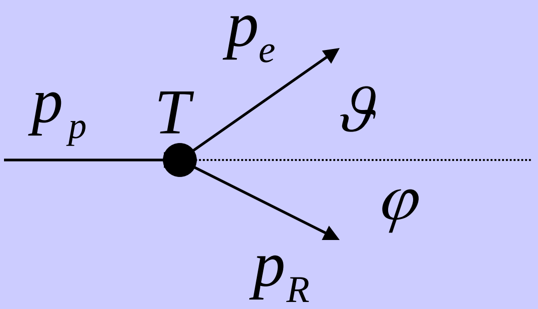
# Energy conservation, Q value

- Radioactive decays $X \rightarrow a + b + \ldots$
  requires $Q > 0$ (exothermic), to be possible due to energy conservation.

- Endothermic process is only allowed in nuclear reactions in case the kinetic energy of reaction partners (projectile energy for fixed target) is sufficient to provide the needed energy for the reaction.

# Q-value equation

Typical observables: momenta
scattering angle
energy




e.g. $p_p$ , $p_e$ , $\vartheta$

Conservation of momentum in reaction plane:

→ longit. $$p_p = p_e \cdot \cos\vartheta + p_R \cos\varphi$$

↑ trans. $$0 = p_e \cdot \sin\vartheta - p_R \sin\varphi$$

# Q-value equation

Solve equation for: $\varphi$

$$p_R \cos\varphi = p_p - p_e \cos\vartheta$$

$$p_R \sin\varphi = p_e \sin\vartheta$$

$$p_R{}^2 = p_p{}^2 + p_e{}^2 - 2 p_p p_e \cos\vartheta$$

use $T = \frac{p^2}{2m}$ non-relativistic

$$Q = T_R + T_e - T_p$$

$$= \frac{1}{2m_R}\left(p_p{}^2 + p_e{}^2 - 2 p_p p_e \cos\vartheta\right) + T_e - T_p$$

$$= T_e\left(1 + \frac{m_e}{m_R}\right) - T_p\left(1 + \frac{m_p}{m_R}\right) - \frac{1}{m_R}\left(2m_p T_p\right)^{1/2}\left(2m_e T_e\right)^{1/2}\cos\vartheta$$

# Q-value equation

Q-value equation:

$$Q = T_e\left(1+\frac{m_e}{m_R}\right) - T_p\left(1-\frac{m_p}{m_R}\right) - \frac{2\sqrt{m_e T_e m_p T_p}}{m_R}\cos\vartheta$$

Equation depends on beam energy, energy of scattered projectile, scattering angle. Masses are often known

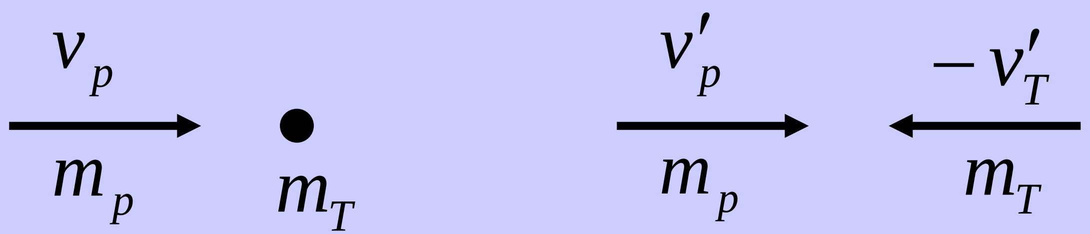
- For very small $T_p$ one may use $T_e=Q\,(m_R/(m_e+m_R))$

- There can be two combinations for $T_p$ and $T_e$ for one scattering angle!
- Threshold energy in laboratory system for endothermic reaction

# Q-value & energy threshold

For $Q<0$ (endothermic) a lowest minimum kinetic threshold energy $T_p$ of the projectile is needed in the laboratory system in order to enable the reaction. This energy is larger than Q value, which is valid vor the CM system. Energy of  center of mass movement has to be taken into account!

Lab. system CM system




$$v'_T = \frac{m_p}{m_T} v'_p$$

$$T' = \frac{1}{2} m_p {v'_p}^2 + \frac{1}{2} m_T {v'_T}^2 = \frac{1}{2} m_p {v'_p}^2 \left(1+\frac{m_p}{m_T}\right) = |Q|$$

Energy balance in CM system

# Q-value & energy threshold

back transformation into laboratory:

$$v_p = v'_p + v'_T$$
$$= v'_p\left(1 + {m_p}/{m_T}\right)$$

Minimum energy:

$$T_p = \frac{1}{2} m_p {v_p}^2$$
$$= \frac{1}{2} m_p {v'_p}^2 \left(1 + {m_p}/{m_T}\right)^2$$
$$= |Q|\left(1 + {m_p}/{m_T}\right)$$

Example:
$\alpha + {}^4He \rightarrow {}^7Li + p$
Q-value: $Q = -17$ MeV
Threshold energy for reaction:
$T_p = 17 \cdot (1+4/4)$ MeV
$= 34$ MeV
Causes large kinetic energy of center of mass (fast moving products after reaction)

$$|Q| = \frac{1}{2} m_p {v'_p}^2 \left(1 + \frac{m_p}{m_T}\right)$$

Threshold energy in laboratory!

# Example

In any nuclear reacting at any incident energy, the total energy (including the kinetic energy and the rest mass-energy given $E=mc^2$), remains invariant, both in the lab system and the centre of mass system.

**Lab system:**

$^{4}He + {}^{60}Ni + E_i \longrightarrow {}^{62}Cu + p + n + E_f$

$E_f = E_R({}^{62}Cu) + E_p + E_n + E_\gamma$

$E_\gamma = {}^{62}Cu^* - {}^{62}Cu$

$Q = M_i - M_f = E_f - E_i$

$M_f = M({}^{62}Cu) + M_p + M_n$

$E_f$ is the total kinetic energy of particles plus energy of gamma-rays

$E_i$ is the incident energy

**CM system**

Q=T(out) –T(in)

T(out) is the total kinetic energy of exit channel of all reaction products in the CM system

T(in) is the total kinetic energy of incident channel in the CM system

**Q-value is the same, in both CM and Lab systems!**

# Summary: Energy conservation, Q value

Relativistic conservation of energy

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Q value:

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$$= T_R + T_e - T_T - T_p = T_{final} - T_{initial}$$

Q-value equation:

$$Q = T_e\left(1 + \frac{m_e}{m_R}\right) - T_p\left(1 - \frac{m_p}{m_R}\right) - \frac{2\sqrt{m_e T_e m_p T_p}}{m_R}\cos\vartheta$$

# Principle of Detailed Balance

$$\underbrace{1+2}_{a} \underset{\sigma_b}{\overset{\sigma_a}{\rightleftarrows}} \underbrace{3+4}_{b}$$

The cross sections $\sigma_a$ and $\sigma_b$ are related through the ‚Principle of Detailed Balance‘;
Based on time reversal invariance: t→-t

Reminder: cross section

$$\sigma = \frac{\text{total number of interactions} \;/\text{sec}}{\text{particle flux} \bullet \text{scattering centers}} = \frac{\dot{N}}{\phi_a \cdot N_b}$$

# Reminder: cross section

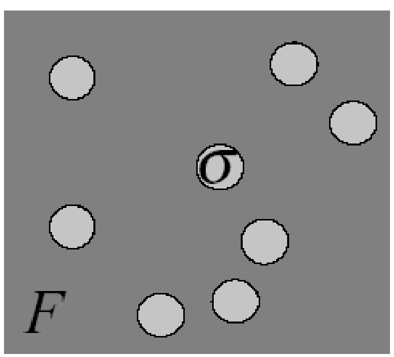
## Wirkungsquerschnitt WQ

WQ ist verknüpft mit der quantenmechanischen Wahrscheinlichkeit, daß in einem Streuexperiment eine bestimmte Wechselwirkung eintritt und ein gewisser Endzustand nach der Streuung vorliegt.

WQ ist abhängig von:
- Energie
- Art der Wechselwirkung
- Streupartner die an der Reaktion beteiligt sind

Anschauliche Interpretation: "Wirksame" Fläche eines Kerns, die zu Reaktion führt.
Grobe Abschätzung: Geometrische Fläche

Einheit: 1 barn (b) = $10^{-28}$ [m$^2$] = $10^{-24}$ [cm$^2$]
1 millibarn (mb)= $10^{-27}$ [cm$^2$]




Geometrische Wahrscheinlichkeit für Reaktion

$$P = \frac{N \cdot \sigma}{F}$$

N Anzahl der Streuzentren
F Fläche
$\sigma$ Wirkungsquerschnitt

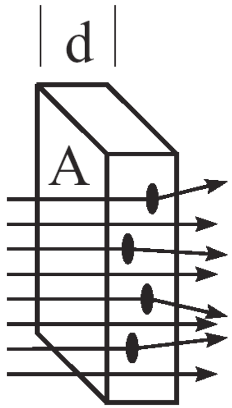
# reminder: total cross section

$$\sigma = \frac{\text{Zahl der Reaktionen/Zeit}}{\text{Strahlteilchen/Zeit} \bullet \text{Streuzentren/Fläche}} = \frac{\dot{N}}{\dot{N}_a \cdot N_b / A} = \frac{\dot{N}}{I_a \cdot N_b^F}$$

Strahl — Target — Strom — Flächen dichte

Die Targetfläche fällt bei der Berechnung heraus, wenn man die Anzahl der Streuzentren in der Targetfolie berechnet.
Voraussetzung: dünne Folie, Streuzentren überlappen nicht bzw schirmen sich nicht ab.

Definitionen:




$n_a$ : Teilchendichte im Strahl

$N_a$ : Teilchenanzahl im Strahl

$v_a$ : Strahlgeschwindigkeit

$n_b$ : Streuzentrendichte im Target

$N_b$ : Streuzentrenanzahl im Target

$A$ : Querschnittsfläche des Targets

$d$ : Dicke des Targets

$\dot{N}_a = I_a$ : Teilchenstrom im Strahl = Teilchen/Zeit

$\Phi_a = \dot{N}_a / A$ : Teilchenfluß = Teilchen/Zeit · Fläche

$N_b = n_b A d$

$N_b^F = N_b / A$ : Flächendichte der Streuzentren im Target

Reaktionsrate im Target :

$$\dot{N} = \Phi_a N_b \sigma_b = I_a N_b^F \sigma_b$$

# Fermis Golden Rule

$$W_{fi} = \frac{2\pi}{\hbar}\left|H_{fi}\right|^2 \cdot \frac{dn}{dE_f} = \frac{2\pi}{\hbar}\left|M\right|^2 \cdot \rho\left(E_f\right)$$

$W_{fi}$ - transition probability from initial state to final state

$H_{fi}$ , $M$ - matrix element
- information about nuclear states
- interaction mechanism

$\rho\left(E_f\right)$ - density of final states
phase space factor

- If the Hamilton operator $\hat{H}$ is hermitian:

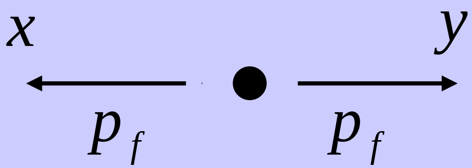
$$M_{if} = \left\langle f \middle| \hat{H} \middle| i \right\rangle = \left\langle i \middle| \hat{H} \middle| f \right\rangle^*$$

$$\left|M_{a\to b}\right|^2 = \left|M_{b\to a}\right|^2 = \left|M\right|^2$$ „Detailed Balance“

- Density of states $$\rho\left(E_f\right) = \frac{4\pi}{h^3} \cdot p_f^2 \cdot \frac{dp_f}{dE_f} \cdot g_f$$

because $T(p,e)R$ is a two body problem with two body kinematics in the centre of mass system (CM) for momenta in final state is only one independent momenta $p_f$

$$\vec{p}_e = -\vec{p}_R$$



$$E_f = \frac{P_f^2}{2m_x} + \frac{P_f^2}{2m_y}$$

$$\frac{dE_f}{dP_f} = \frac{P_f}{m_x} + \frac{P_f}{m_y} = P_f\left(\frac{m_x + m_y}{m_x m_y}\right)$$

$$= P_f \cdot \frac{1}{m_f} = v_f \frac{m_f}{m_f}$$

$$= v_f$$

$m_f$ - reduced mass in final channel

$v_f$ - relative velocity of final particles after reaction

$$\text{particle flux} = \text{particle density} \times \text{velocity}$$

$$= n_a \cdot v_a$$

$$= \frac{N_a}{V} \cdot v_a$$

$$= \frac{N_a}{A} \cdot \frac{1}{s} = \frac{\dot{N}_a}{A} = \frac{\text{particle number}}{\text{area} \cdot \text{time}}$$

For one particle per unit volume (considering only 1 incident particle)

$$= \frac{1}{V} \cdot v_a = v = \text{flux}$$

$$\hat{=} \text{ relative velocity in initial state}$$