In general: consider reaction **1+2 → 3+4** in a plasma

Nuclei of type **1** and type **2** are destroyed via the nuclear reaction **1+2** and nuclei of type **3** and type **4** **produced**.

The reaction rate is determined by:

- Particle density (number of particles/volume)
- Kinectic energy/ relative velocity of reaction partners
- Reaction cross section which depends on energy/velocity in CM system

N_i = density of interacting particles [number/volume]= [number/ m^3]
v = relative velocity [m/s]
$\phi(v)$ = velocity distribution in plasma
$\sigma(v)$ = reaction cross section [m^2]

reaction rate depends on two factors :

(i) Effective area, which contributes to reaction rate, is proportional

$\sigma(v) \cdot N_1$

(ii) particle flux

$v \cdot N_2$

reaction rate per volume is proportional $[1/(\text{time} \cdot \text{volume})$ units $1/[\text{s} \cdot \text{m}^3]]$

$r = N_1 N_2 \sigma(v) v$

velocity is given with a certain probability distribution :

$$\int_0^\infty \Phi(v) dv = 1$$

reaction rate per particle pair and volume

$$\langle \sigma v \rangle = \int_0^\infty \Phi(v) v \sigma(v) dv$$

N_i = density of interacting particles [number/ m^3]
v = relative velocity [m/s]
$\phi(v)$ = velocity distribution in plasma
$\sigma(v)$ = reaction cross section [m^2]

for exothermal reaction; integration starts from $v = 0$ until $v = \infty$

for endothermal reaction; integration starts from $v \propto Q^{1/2}$ until $v = \infty$

totale reaction rate

$$r = \frac{1}{1+\delta_{12}} N_1 N_2 \langle \sigma v \rangle$$

Kronecker delta applies for identical particles to avoid double counting.

Consider reaction: **1+2 → 3**

Nuclei of type **1** are destroyed via the capture reaction with type **2** nuclei;

type **3** nuclei are produced.

average live time of type **1** nuclei in stellar environment is given by differential equation:

$$\left(\frac{dN_1}{dt}\right)_2 = -\lambda N_1 = -\frac{1}{\tau} N_1$$

Mean lifetime

Decay constant

or

$$\left(\frac{dN_1}{dt}\right)_2 = -(1+\delta_{12})\, r = -N_1 N_2 \langle \sigma v \rangle$$

Kronecker symbol disappears,
often the following equations are given:

$$= -N_1 \rho N_A \frac{X_2}{A_2} \langle \sigma v \rangle = -N_1 \rho N_A Y_2 \langle \sigma v \rangle$$

$$r = \frac{1}{1+\delta_{12}} N_1 N_2 \langle \sigma v \rangle$$

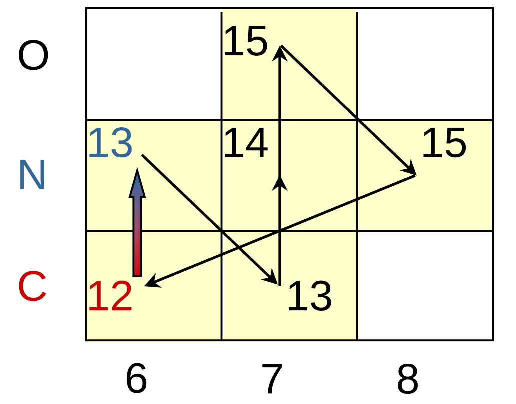
$$N_i = \rho N_A \frac{X_i}{A_i} = \rho N_A Y_i$$

number density	N_i
Matter density	ρ
Avogadro‘s number	N_A
Mass fraction	X_i
Atomic mass	A_i
Mol fraction	Y_i

$$\frac{dN_1}{dt} = -N_1\lambda = -\frac{1}{\tau}N_1 = -(1+\delta_{12})\,r$$

$$= -N_1 Y_2 \rho N_A \langle\sigma v\rangle$$

example

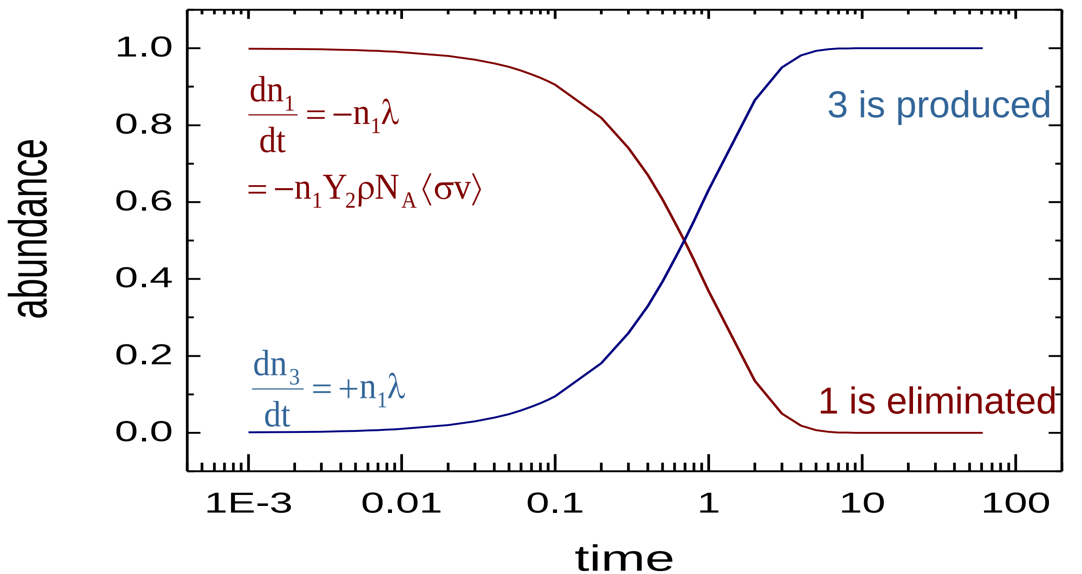


Definition:
Mean lifetime of nuclei 1 against destruction by 2:

$$\tau = \frac{1}{\lambda} = \frac{1}{Y_2 \rho N_A \langle\sigma v\rangle}$$

Lifetime of nuclei 1 depends on:

- Number of destructive nuclei 2
- Velocity distribution
- Cross section



Reaction rate, energy production

- reaction rate r
  (number of reactions per time and volumen)

$$r = \frac{1}{1+\delta_{pT}} N_p N_T \langle \sigma v \rangle$$

$$\langle \sigma v \rangle = \int \sigma(v) \cdot v \cdot \phi(v) dv$$

Key quantities

N_i = density of interacting nuclei (number/volume)
v = relative velocity
$\phi(v)$ = velocity distribution in plasma
$\sigma(v)$ = reaction cross section

- energy production rate ε

$$\varepsilon = r\, Q/\rho$$

Literatur: Cauldrons in the Cosmos, chapter 3
C. E. Rolfs, W. S. Rodney
The University of Chicago Press

- energy production rate ε
  (energy per time and mass)

example

Hydrogen burning

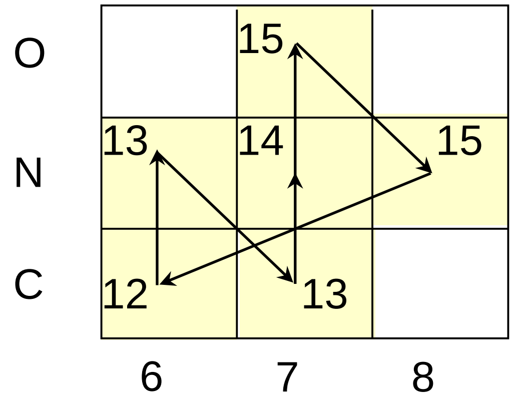
netto result: $4p \rightarrow {}^{4}He + 2e^{+} + 2\nu + Q_{eff}$

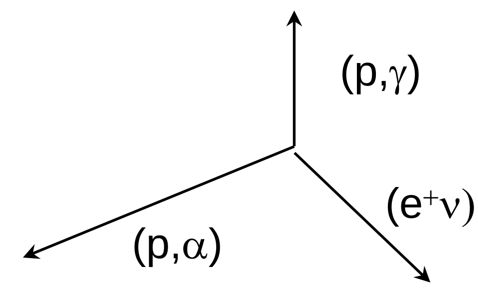
$Q_{eff} = 26.73$ MeV

nucleo synthesis

Energy production

CNO Zyklus





${}^{12}C(p,\gamma){}^{13}N(e^{+}\nu){}^{13}C(p,\gamma){}^{14}N(p,\gamma){}^{15}O(e^{+}\nu){}^{15}N(p,\alpha){}^{12}C$

cycle is limited by β decay of ${}^{13}N$ (t ~ 10 min) and ${}^{15}O$ (t ~ 2 min)

C, N, O isotopes act as catalysts

changes in stellare conditions ⇒ modification of energy production and nucleo synthesis

REACTION RATES are needed in function of velocity~temperature of stellar environment to determine ENERGY PRODUCTION.

Stellare reaction rate

$$\langle \sigma v \rangle = \int \sigma(v)\phi(v) v dv$$

Depends on: a) velocity distribution
b) cross section

velocity distribution

Interacting nuclei in plasma are in thermal equilibrium at temperature T,

Assume a non-degenerated and non-relativistic plasma

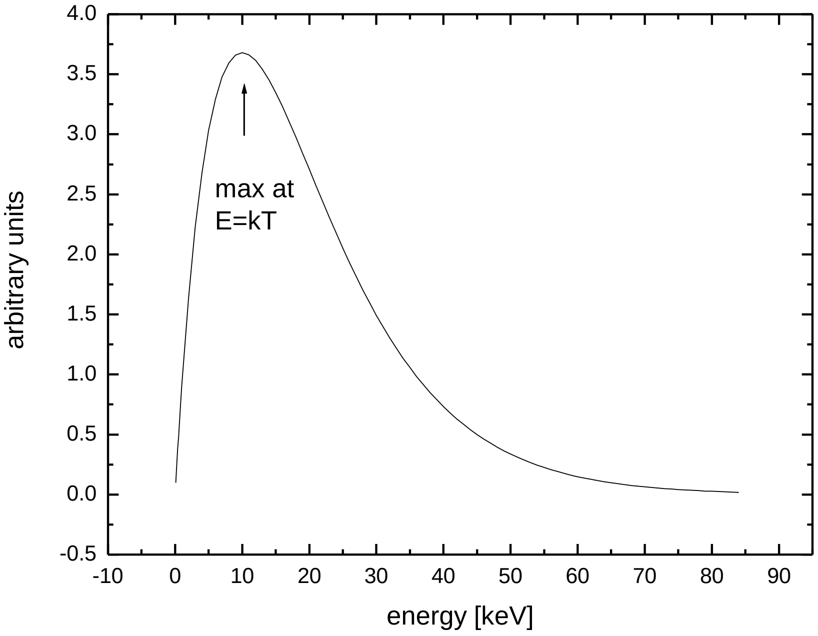
⇒ Maxwell-Boltzmann velocity distribution

$$\phi(v) = 4\pi \left(\frac{\mu}{2\pi kT}\right)^{3/2} v^2 \exp\left(-\frac{\mu v^2}{2kT}\right)$$

with $\mu = \frac{m_p m_T}{m_p + m_T}$ reduced mass

v = relativ velocity, $E = 1/2\ \mu v^2$

$$\langle \sigma v \rangle = \left(\frac{8}{\pi\mu}\right)^{1/2} \frac{1}{(kT)^{3/2}} \int_0^\infty \sigma(E) \cdot E \cdot \exp\left(-\frac{E}{kT}\right) dE$$



$kT \sim 8.6 \times 10^{-8}\ T[K]$ keV

example: sun $T \sim 15 \times 10^6$ K oder $T_6 = 15$ ⇒ $kT \sim 1.3$ keV

Cross section

There is no theory which allows to determine the reaction cross section precisely enough!

Depends critically on the following reasons:

- properties and structure of nuclei
- reaction mechanism

Cross sections may vary by orders of magnitude depending in interaction!

Examples:

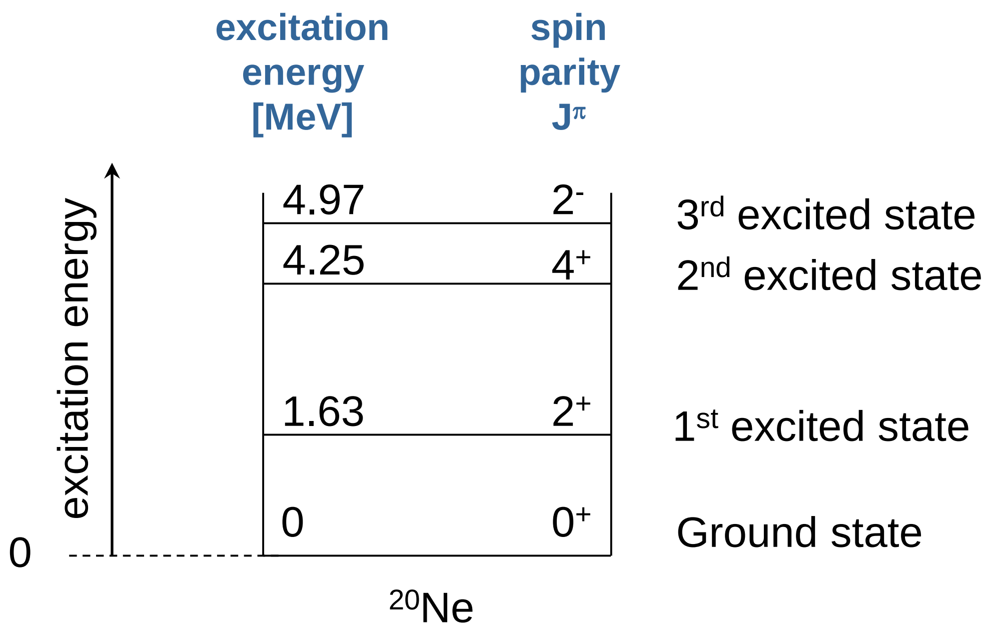
Reaction	interaction/force	σ (barn)	E_{proj} (MeV)
$^{15}N(p,\alpha)^{12}C$	Strong interact.	0.5	2.0
$^{3}He(\alpha,\gamma)^{7}Be$	Strong+E.M.	10^{-6}	2.0
$p(p,e^{+}\nu)d$	weak WW	10^{-20}	2.0

1 barn = 10^{-24} cm^2 = 100 fm^2

Knowledge from **experiments** and t**heory is needed** to determine the stellare reaction rate.

reminder: nucleons inside atomic nucleus occupy quantized levels (→ shell model) with fixed excitation energy

example



Nucleus in excited state decays typically via **γ, p, n** or **α emission** with characteristic **Lifetime τ** with a corresponding **width Γ** in excitation energy.

$$\Gamma = \frac{\hbar}{\tau}$$

Heisenbergs' uncertainty relation

Lifetime of individual decay branches or channels is given by the **partial width**

Γ_γ, Γ_p, Γ_n and Γ_α with $\Gamma = \sum \Gamma_i$

Reaction mechanism:

I. direct reaction

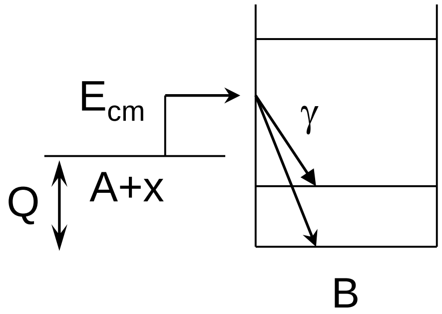
II. resonant reaction

I. direct process

One-step process

Direct transition into bound state

example: radiative capture A(x,γ)B



$$\sigma_\gamma \propto \left| \langle B | H_\gamma | A + x \rangle \right|^2$$

H_γ = electro-magnetic operator describes transition

- reaction cross section is proportional to one single matrix element
- happens at al projectile energies
- uniformly, continuous energy dependence of cross section

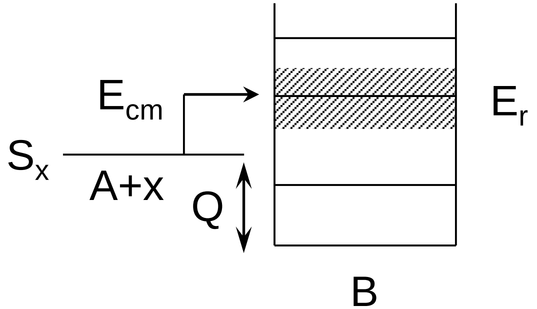
other direct processes: stripping, pickup, charge exchange, Coulomb excitation

II. resonant process

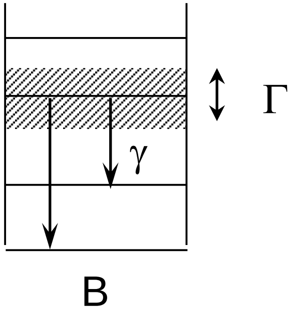
Two-step process

example: **resonant radiativ capture A(x,γ)B**

1. Compound-nucleus formation
(in one unbound state)



2. Compound nucleus decay
(in lower lying states)



$$\sigma_\gamma \propto \left|\langle E_f | H_\gamma | E_r \rangle\right|^2 \; \left|\langle E_r | H_B | A + x \rangle\right|^2$$

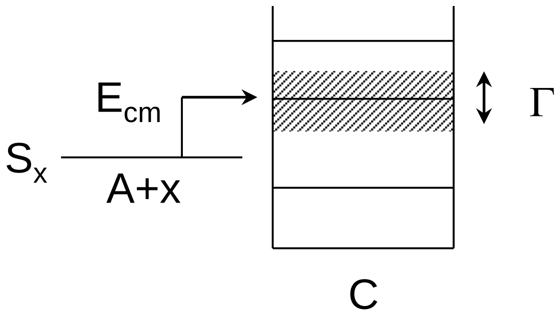
Compound nucleus decay — probability $\propto \Gamma_\gamma$

Compound nucleus formation — probability $\propto \Gamma_x$

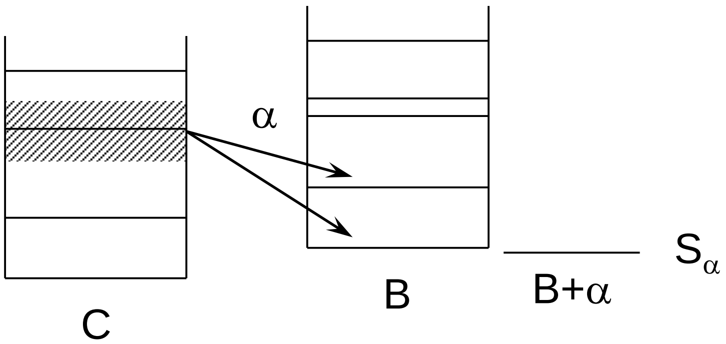
- Reaction cross section is proportional to two matrix elements
- strong enhancement close to energy $E_{cm} \sim E_r - Q$
- strong energy dependence of cross section

Energy in entrance channel ($Q+E_{cm}$) must be close in excitation energy E_r of resonant state. However, every excited state has a width $\Rightarrow$ cross section from tails of distribution example: **resonant reaction A(x,α)B**

1. Compound-nucleus formation
(in one unbound state)



2. Compound nucleus decay
(in lower lying states via particle emission)



$$\sigma_\alpha \propto \left|\left\langle B+\alpha \middle| H_\alpha \middle| E_r \right\rangle\right|^2 \left|\left\langle E_r \middle| H_x \middle| A+x \right\rangle\right|^2$$

Compound nucleus decay
probability $\propto \Gamma_\gamma$

Compound nucleus formation
probability $\propto \Gamma_x$

Energy in entrance channel ($Q+E_{cm}$) must be close in excitation energy E_r of resonant state.
However, every excited state has a width $\Rightarrow$ cross section from tails of distribution

Cross section for direct reactionc

- Charged particles
- Neutrons

Cross section for direct reaction

example: direct capture A + x → B + γ

$$\sigma = \pi \lambda_x^2 \left| \langle B | H | x + A \rangle \right|^2 P_\ell(E)$$

"geometrical factor"
de Broglie wave length of projectile

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}$$

matrix element contains nuclear properties of interaction

- Penetrability/ Transmission probability of projectile to interact with target nucleus
- Depends on angular momentum of projectil ℓ and energy E

$$\sigma = \frac{1}{E} \cdot P_\ell(E) \cdot S(E)$$

σ = (strong energy dependence) x (weak energy dependence)

S(E) = astrophysical factor

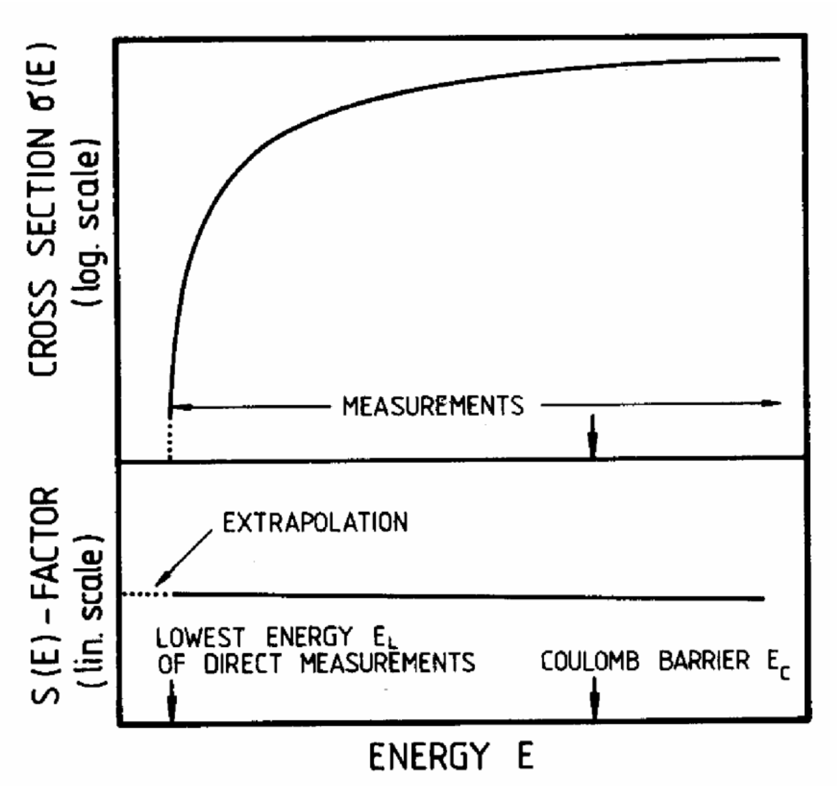
Contains nuclear physics of reaction, matrix element, wave functions, operator

Needed: penetrability $P_\ell(E)$

Transmission Probability depends on:

- Coulomb barrier (only charged particles)
- centrifugal barrier (for neutrons and charged particles)

Astrophysical S-factor



"astro physical S factor" contains detailed information on nuclear structure

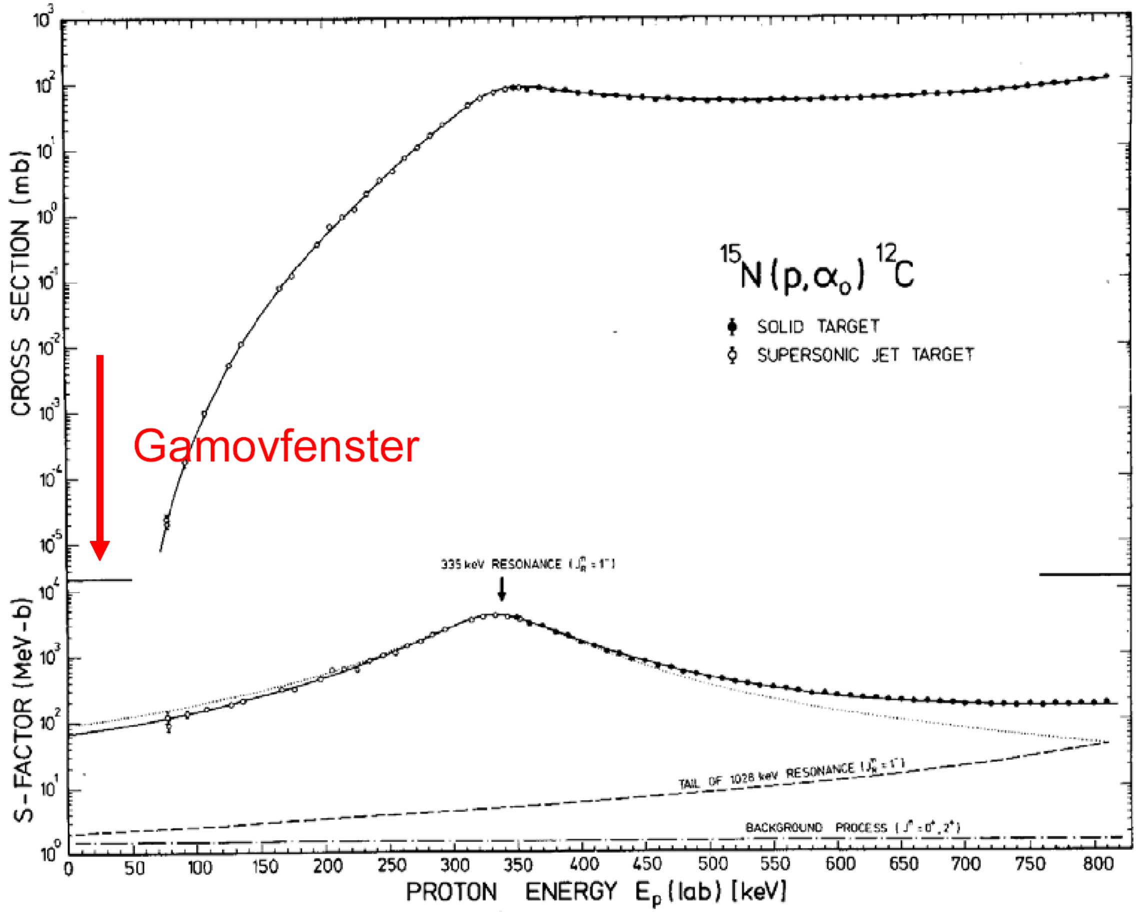
$$\sigma(E) = \frac{S(E)}{E} e^{-b/\sqrt{E}}$$

Relevant energy region for astro physics
Is very low, typicalle at the limit and below
measurable energy for nuclear reaction.

In some cases S(E) can be extrapolated.

Rolfs & Rodney p. 156

Astrophysical S-factor



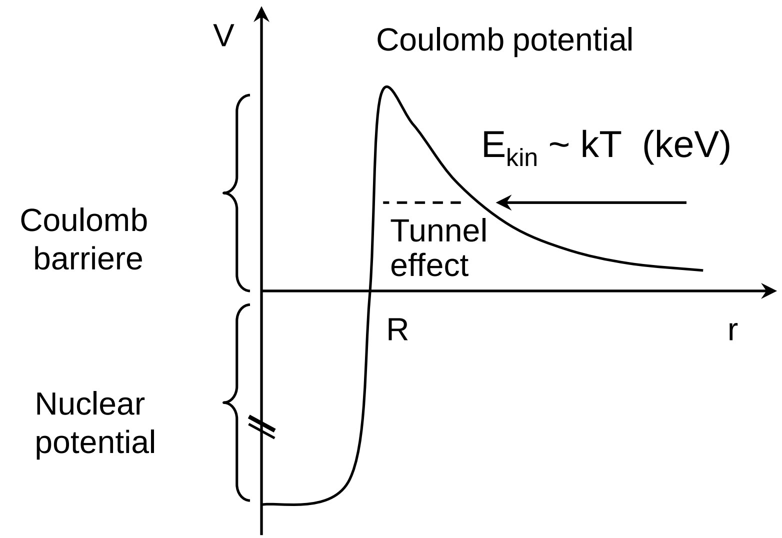
Rolfs & Rodney p. 158

Typically data for astrophysical processes are given by S(E).
In some cases S(E) varies strongly with energy.

Tunneling Gamow factor

Projectil and target with charges Z_1 and Z_2

$$V_C = \frac{Z_1 Z_2 e^2}{R}$$



in units (MeV, amu, fm)

$$V_C[\mathrm{MeV}] = 1.44 \frac{Z_1 Z_2}{R[\mathrm{fm}]} \approx 1.2 \frac{Z_1 Z_2}{\left(A_1^{1/3} + A_2^{1/3}\right)}$$

example: $^{12}C(p,\gamma)$ V_C= 3 MeV

average kinetic energy in stellar plasma: kT ~ 1-100 keV!

⇒ Fusion reaction of two charged particles well below Coulomb barrier

⇒ Transmission probability is determined by tunnel effect

for E<<V_C and no angular momentum transfer tunnelling is given by:

$$P(E) \propto \exp(-2\pi\eta)$$

$$\eta = \frac{Z_1 Z_2 e^2}{\hbar v}$$

$$2\pi\eta = 31.29 Z_1 Z_2 \left(\frac{\mu}{E}\right)^{1/2}$$

Gamow factor

η Sommerfeld parameter

with E- keV and μ- amu