Reminder: cross section for direct reaction

example: direct capture A + x → B + γ

$$\sigma = \pi \lambda_x^2 \left| \langle B | H | x + A \rangle \right|^2 P_\ell(E)$$

"geometrical factor"
de Broglie wave length
of projectile

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}$$

matrix element
contains nuclear
properties of
interaction

- Penetrability/ Transmission
probability of projectile to interact
with target nucleus
- Depends on angular momentum
of projectil ℓ and energy E

$$\boxed{\sigma = \frac{1}{E} \cdot P_\ell(E) \cdot S(E)}$$

σ = (strong energy dependence) x (weak energy dependence)

S(E) = astrophysical factor

Contains nuclear physics of reaction, matrix element, wave functions, operator

Needed: penetrability $P_\ell(E)$

Transmission Probability depends on:

- Coulomb barrier (only charged particles)
- centrifugal barrier (for neutrons and charged particles)

Reminder: Gamow peak

$$\langle\sigma v\rangle = \int \sigma(v)\phi(v)v\,dv = \int \sigma(E)\exp(-E/kT)E\,dE$$

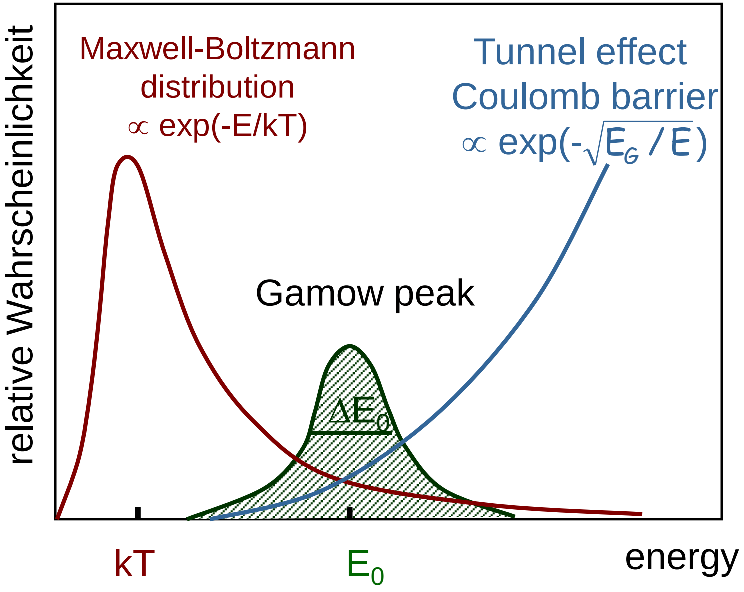
and substitution for σ: $$\langle\sigma v\rangle \propto \int S(E)\exp\left(-\frac{E}{kT}-\frac{b}{\sqrt{E}}\right)dE$$

maximum of reaction rate at E_0: $$\frac{d}{dE}\left[\exp\left(-\frac{E}{kT}-\frac{b}{\sqrt{E}}\right)\right]=0$$

Gamow peak

$$E_0 = \left(\frac{bkT}{2}\right)^{2/3} = 1.22\left(Z_1^2 Z_2^2 \mu\right)^{1/3} T_6^{2/3} \quad \text{keV}$$

$$\Delta E = \frac{4}{\sqrt{3}}\sqrt{E_0 kT} = 0.749\left(Z_1^2 Z_2^2 \mu\right)^{1/6} T_6^{5/6} \quad \text{keV}$$

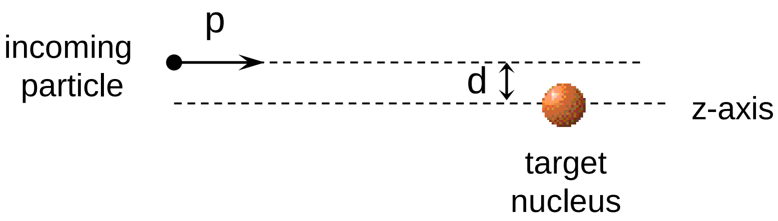


E_0 = relevant energy for astrophysics >> kT

Remark Gamow energy depends on **reaction** and **temperature**

Angular momentum

classical physics:



p = projectile momentum
d = impact parameter

Angular momentum of incoming particle L=p x d

Orbital angular momentum is conserved for central potential

⇒ For higher angular momentum transfer the linear momentum p has to be larger for same d

quantum mechanics:

(discrete values) $L = \sqrt{\ell(\ell+1)}\hbar$

with parity of wave function: $\pi = (-1)^{\ell}$

$\ell = 0$	s - wave
$\ell = 1$	p - wave
$\ell = 2$	d - wave
…	

Orbital angular momentum is conserved for central potential
⇒ finite orbital angular momentum implies “orbital angular momentum energy barrier” V_ℓ

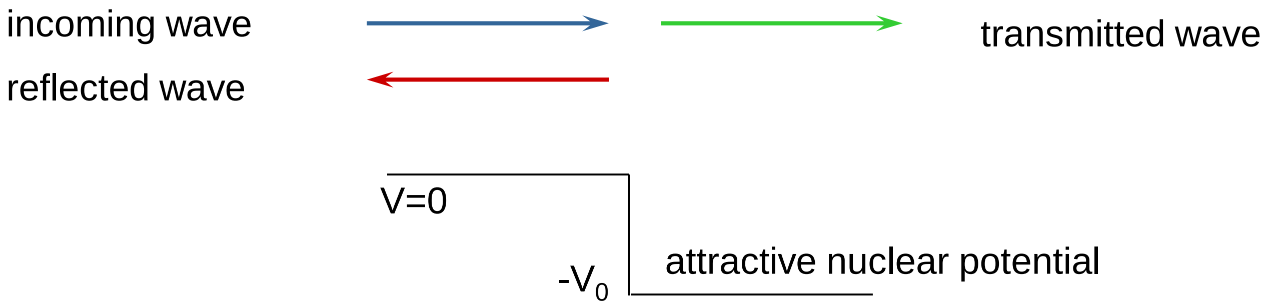
$$\boxed{V_\ell = \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}}$$

μ = reduced mass of projectile-target system
r = radial distance from target nucleus center

Transmission probability

Simplest case: s-wave neutrons $\Rightarrow$ $V_\ell = 0$ and Coulomb potential $V_C = 0$

Discontinuity of potential causes partial reflection of incoming wave



incoming wave

reflected wave

transmitted wave

V=0

$-V_0$

attractive nuclear potential

Transmission probability:

$$P_\ell \propto E^{1/2} \quad \text{for } \ell = 0 \qquad \sigma \propto \frac{1}{\sqrt{E}} = \frac{1}{v}$$

$$P_\ell \propto E^{1/2+\ell} \quad \text{for } \ell \neq 0 \qquad \sigma \propto E^{\ell-1/2}$$

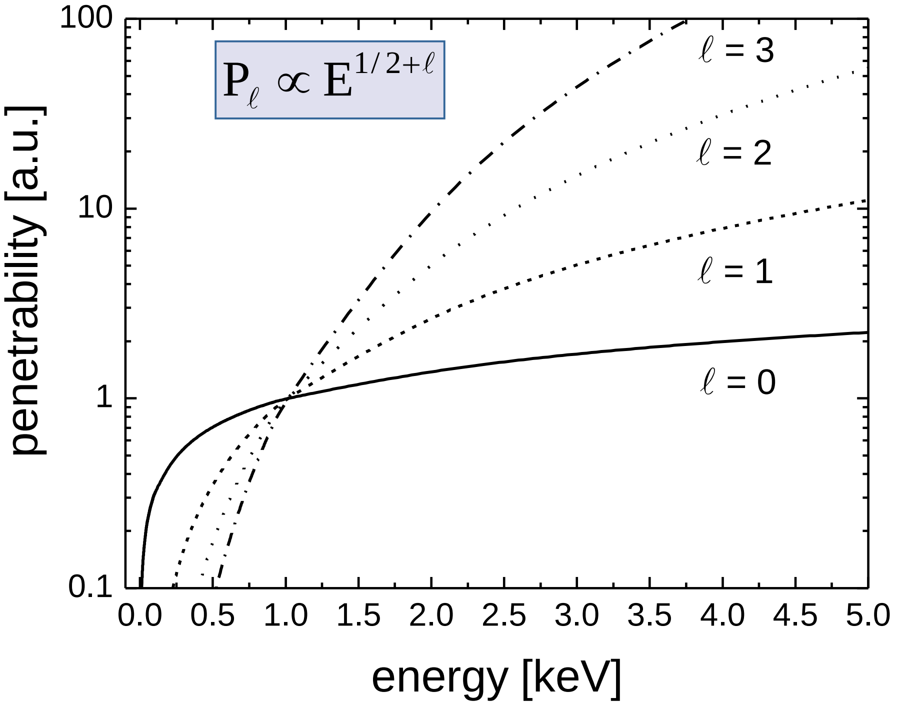
consequences: **s wave:** neutron capture is dominating usually at **low energies** (exception: hinderance due to selection rule)

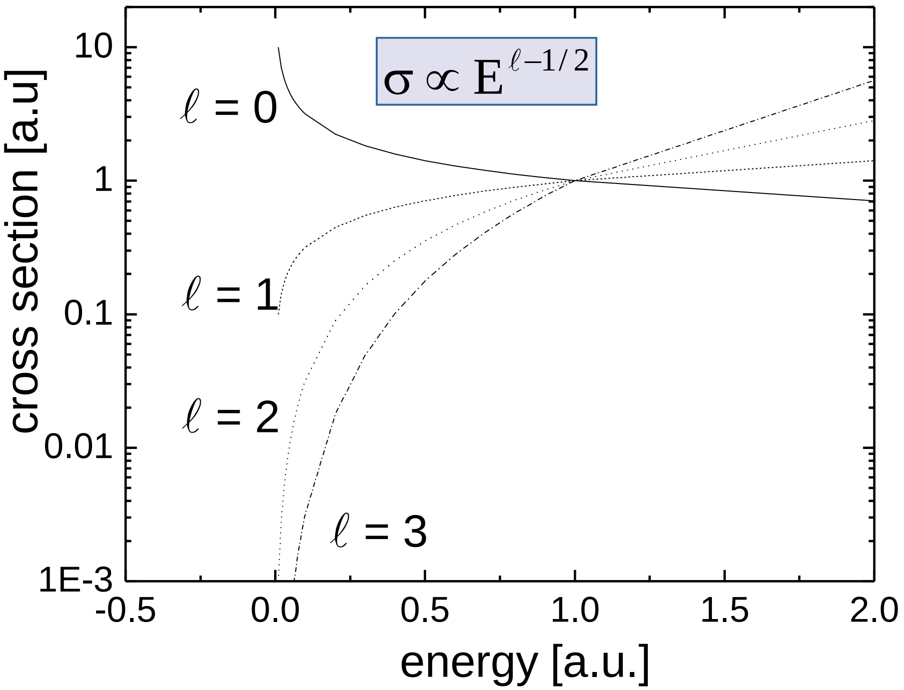
higher ℓ values: neutron capture is only possible at **higher energies** relevant (or when ℓ=0 capture is suppressed)

Angular momentum: pentrability & cross section

ℓ dependence of penetrability through centrifugal barrier

ℓ dependence of neutron capture cross section





low ℓ values dominate reaction rate at low energies

Cross section is strongly reduced at lower energies (angular momentum barriere)

Remark: arbitrary scale between ℓ values

neutron capture

$$\langle\sigma v\rangle \approx \int \sigma(v)\phi(v) v dv \approx \int \sigma(E)\exp(-E/kT)E dE$$

s-wave neutron capture

relevant energy region

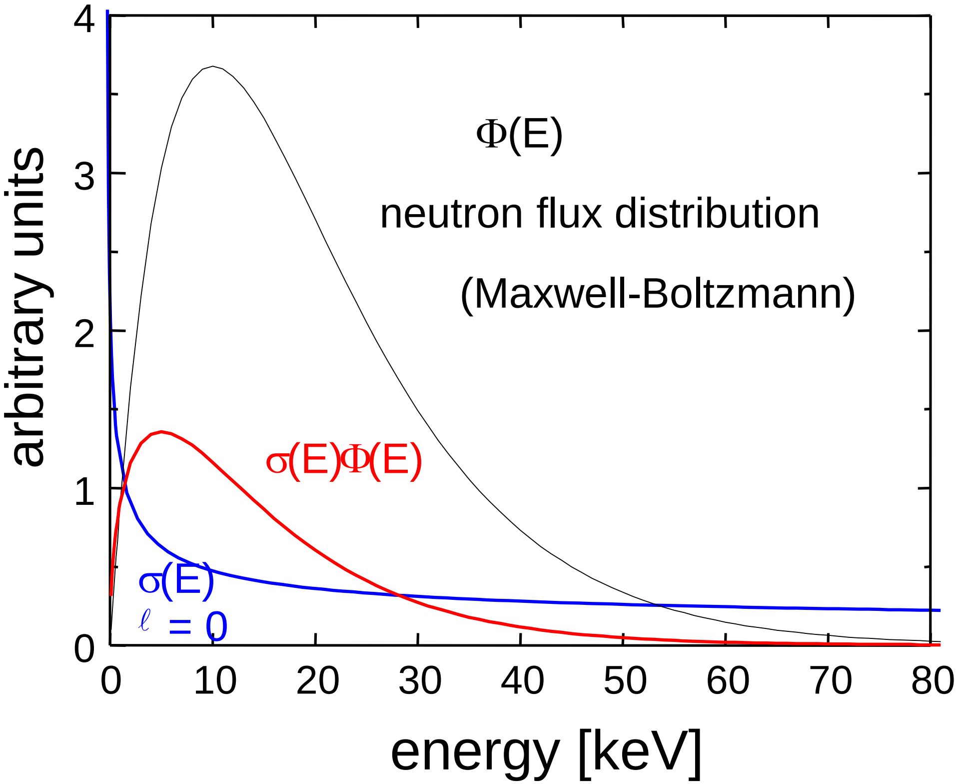
E ~ kT

$$\sigma \propto \frac{1}{\mathrm{v}}$$

$$\boxed{\sigma\mathrm{v} = \mathrm{const} = \langle\sigma\mathrm{v}\rangle}$$

Stellare Reaktionsrate

$$\langle\sigma\mathrm{v}\rangle = \mathrm{v}_T\sigma_{th}$$



σ_{th} = measured cross section for thermal neutrons

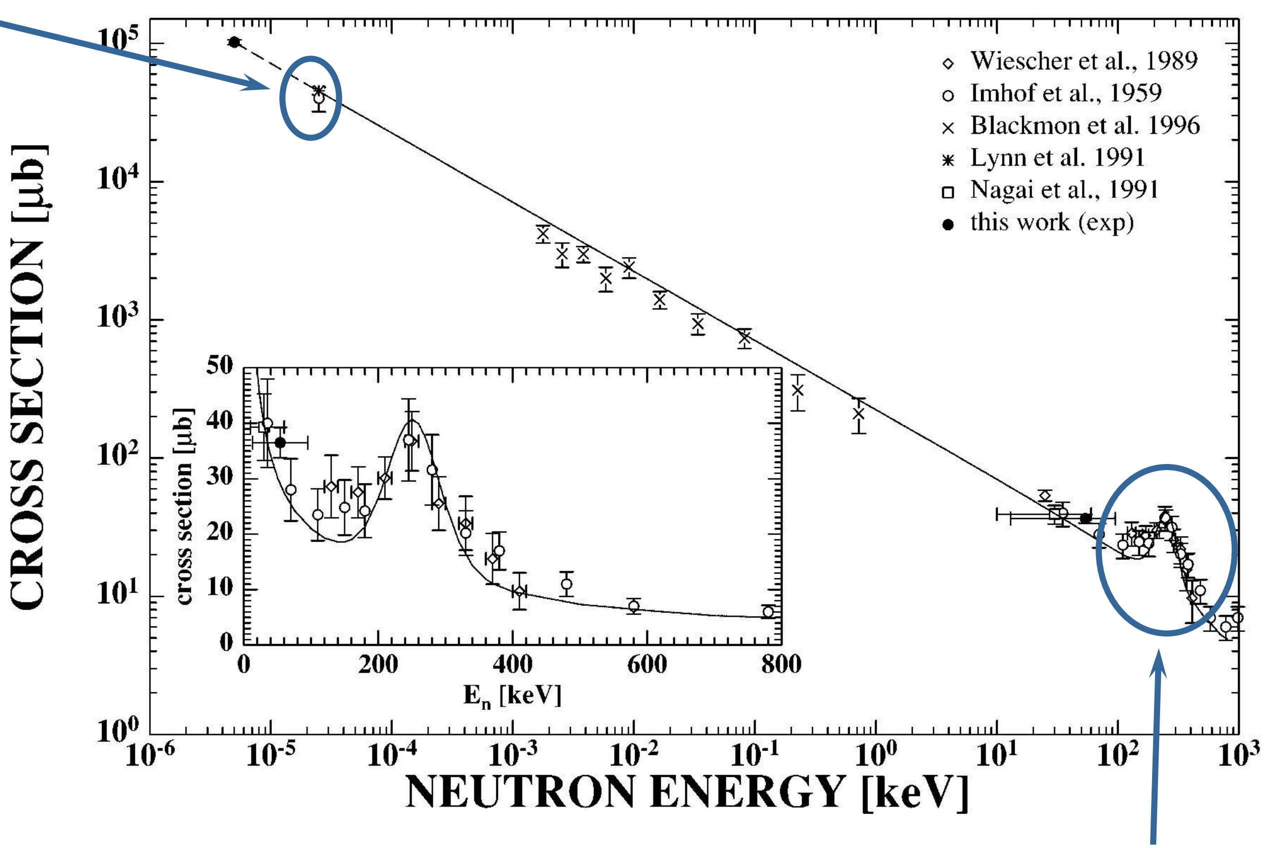
$$\mathrm{v}_T = \sqrt{\frac{2kT}{\mu}}$$ most probable velocity $E_{cm} = kT$

s-wave neutron capture

$$\sigma \propto \frac{1}{\sqrt{E}} = \frac{1}{v}$$

thermal cross section
$<\sigma>$ = 45.4 mb

example: $^{7}Li(n,\gamma)^{8}Li$



Deviation from 1/v behavior due
to resonant contribution

cross section for resonant reactions

for a single isolated resonance:

resonant cross section given by **Breit-Wigner expression**

$$\boxed{\sigma(E) = \pi\lambdabar^2 \frac{2J+1}{(2J_1+1)(2J_T+1)} \frac{\Gamma_1\Gamma_2}{(E-E_r)^2+(\Gamma/2)^2}}$$

for reaction: $1 + T \rightarrow C \rightarrow F + 2$

geometrical factor

$\propto 1/E$

spin factor

J = spin of CN's state

J_1 = spin of projectile

J_T = spin of target

strongly energy-dependent term

Γ_1 = partial width for decay via emission of particle 1

= probability of compound formation via entrance channel

Γ_2 = partial width for decay via emission of particle 2

= probability of compound decay via exit channel

Γ = total width of compound's excited state

$= \Gamma_1 + \Gamma_2 + \Gamma_\gamma + \ldots$

E_r = resonance energy

what about penetrability considerations? $\Rightarrow$ look for energy dependence in partial widths!

partial widths are NOT constant but energy dependent!

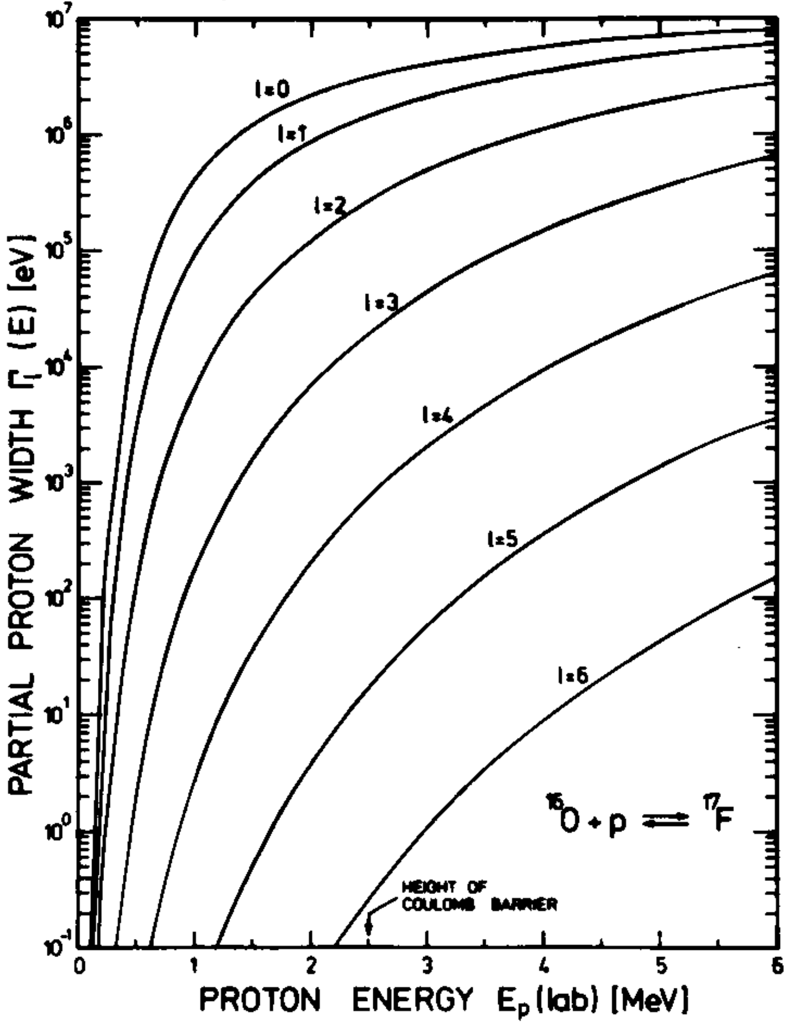
particle widths

$$\boxed{\Gamma_1 = \frac{2\hbar}{R} P_\ell(E_1)\theta_\ell^2}$$

θ_ℓ = “reduced width” (contains nuclear physics info)
P_ℓ gives strong energy dependence
R radius of nuclear potential

example: **$^{16}O(p,\gamma)^{17}F$**

energy dependence of proton partial width Γ_p as function of ℓ



particle partial widths have approximately same energy dependence as penetrability function seen in direct reaction processes

$$\langle \sigma v \rangle = \int \sigma(v)\phi(v)v\,dv = \int \sigma(E)\exp(-E/kT)E\,dE$$

$\downarrow$

here Breit-Wigner cross section

$$\sigma(E) = \pi \bar{\lambda}^2 \frac{2J+1}{(2J_1+1)(2J_T+1)} \frac{\Gamma_1 \Gamma_2}{(E-E_r)^2 + (\Gamma/2)^2}$$

if compound nucleus has an exited state (or its wing) in this energy range

$\Rightarrow$ RESONANT contribution to reaction rate (if allowed by selection rules)

typically:

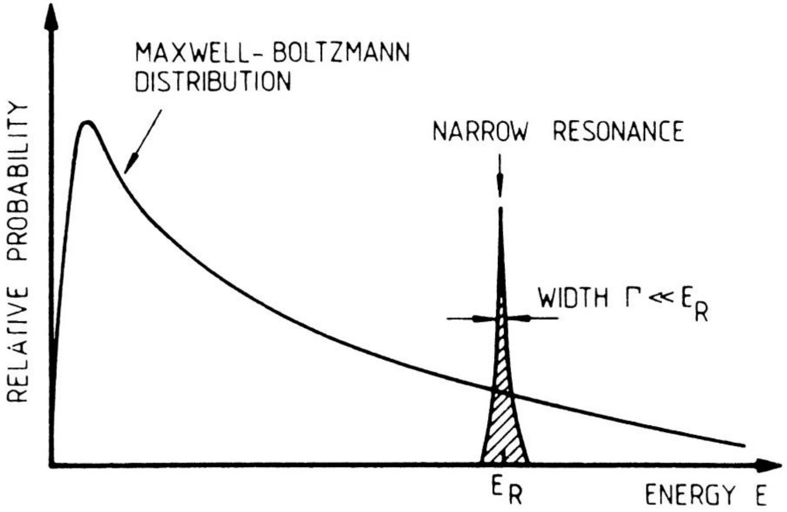
- resonant contribution dominates reaction rate
- reaction rate critically depends on resonant state properties

two simplifying cases:

- narrow (isolated) resonances
- broad resonances

reaction rate for:

- **narrow resonances**



$\Gamma << E_R$

- resonance must be **near** relevant energy range ΔE_0 to contribute to stellar rate
- MB distribution assumed **constant** over resonance region
- partial widths also **constant**, i.e. $\Gamma_i(E) \cong \Gamma_i(E_R)$

reaction rate for a single narrow resonance

$$\boxed{\langle\sigma v\rangle_{12} = \left(\frac{2\pi}{\mu_{12} kT}\right)^{3/2} \hbar^2 (\omega\gamma)_R \exp\left(-\frac{E_R}{kT}\right)}$$

NOTE

exponential dependence on energy means:

- rate strongly dominated by low-energy resonances (**$E_R \rightarrow kT$**) if any
- small uncertainties in E_R (even a few keV) imply large uncertainties in reaction rate

$$\boxed{\langle\sigma v\rangle_{12} = \left(\frac{2\pi}{\mu_{12}kT}\right)^{3/2} \hbar^2 (\omega\gamma)_R \exp\left(-\frac{E_R}{kT}\right)}$$

rate entirely determined by “**resonance strength**” $\omega\gamma$ and **energy of the resonance E_R**

resonance strength
(= integrated cross section over resonant region)

$$\boxed{\omega\gamma = \frac{2J+1}{(2J_1+1)(2J_T+1)}\frac{\Gamma_1\Gamma_2}{\Gamma}}$$

(Γ_i values at resonant energies)

statistical factor: $$\boxed{\omega = \frac{2J+1}{(2J_1+1)(2J_T+1)}}$$

often $\Gamma = \Gamma_1 + \Gamma_2$

width ratio: $$\boxed{\gamma = \frac{\Gamma_1\Gamma_2}{\Gamma}}$$

$$\boxed{\begin{array}{l} \Gamma_1 << \Gamma_2 \longrightarrow \Gamma \approx \Gamma_2 \longrightarrow \frac{\Gamma_1\Gamma_2}{\Gamma} \approx \Gamma_1 \\ \Gamma_2 << \Gamma_1 \longrightarrow \Gamma \approx \Gamma_1 \longrightarrow \frac{\Gamma_1\Gamma_2}{\Gamma} \approx \Gamma_2 \end{array}}$$

experimental info needed:

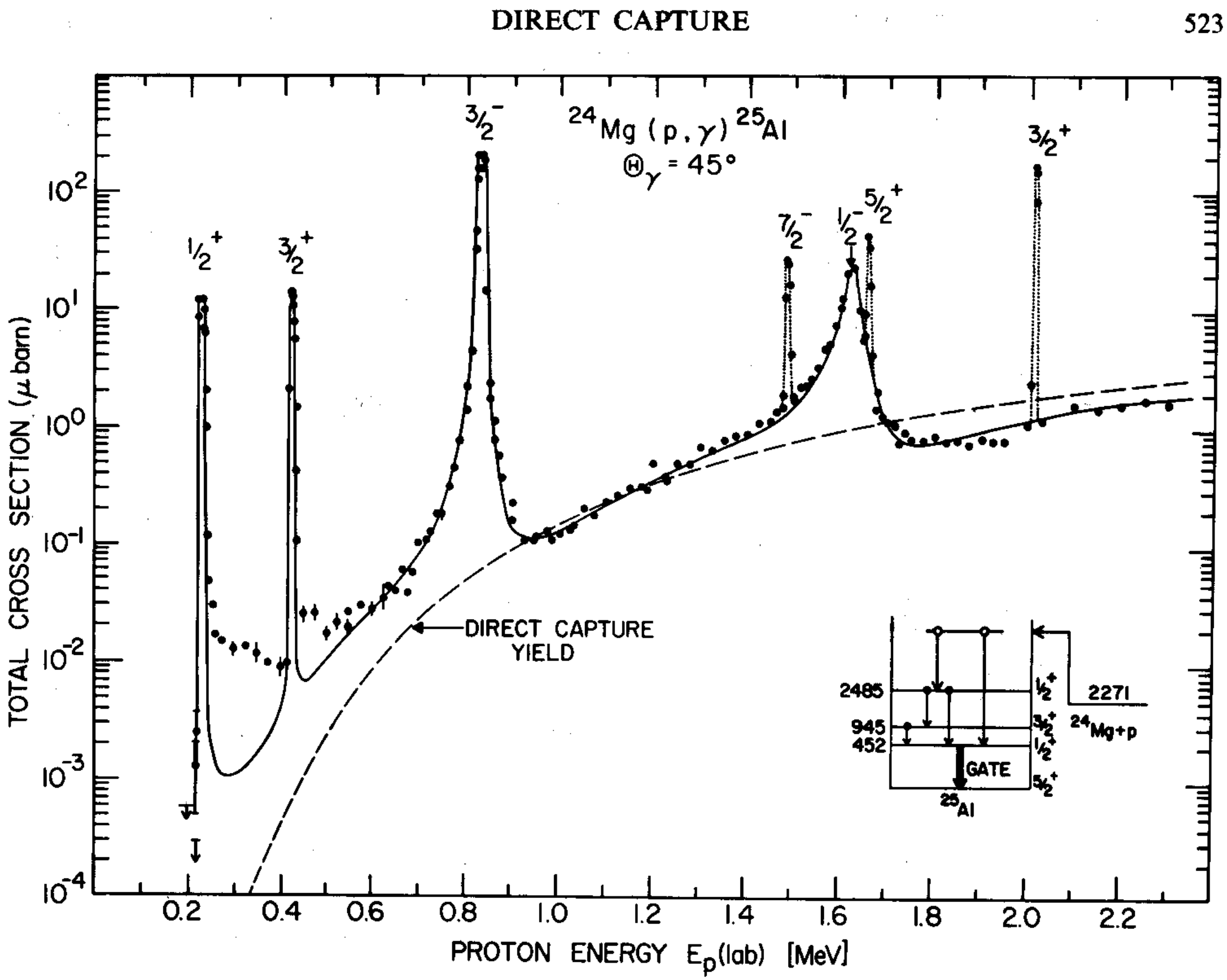
reaction rate is determined by the **smaller** width !

- partial widths Γ_i
- spin **J**
- energy E_R

note: for many unstable nuclei most of these parameters are **UNKNOWN**!

example: ^{24}Mg(p,γ)^{25}Al

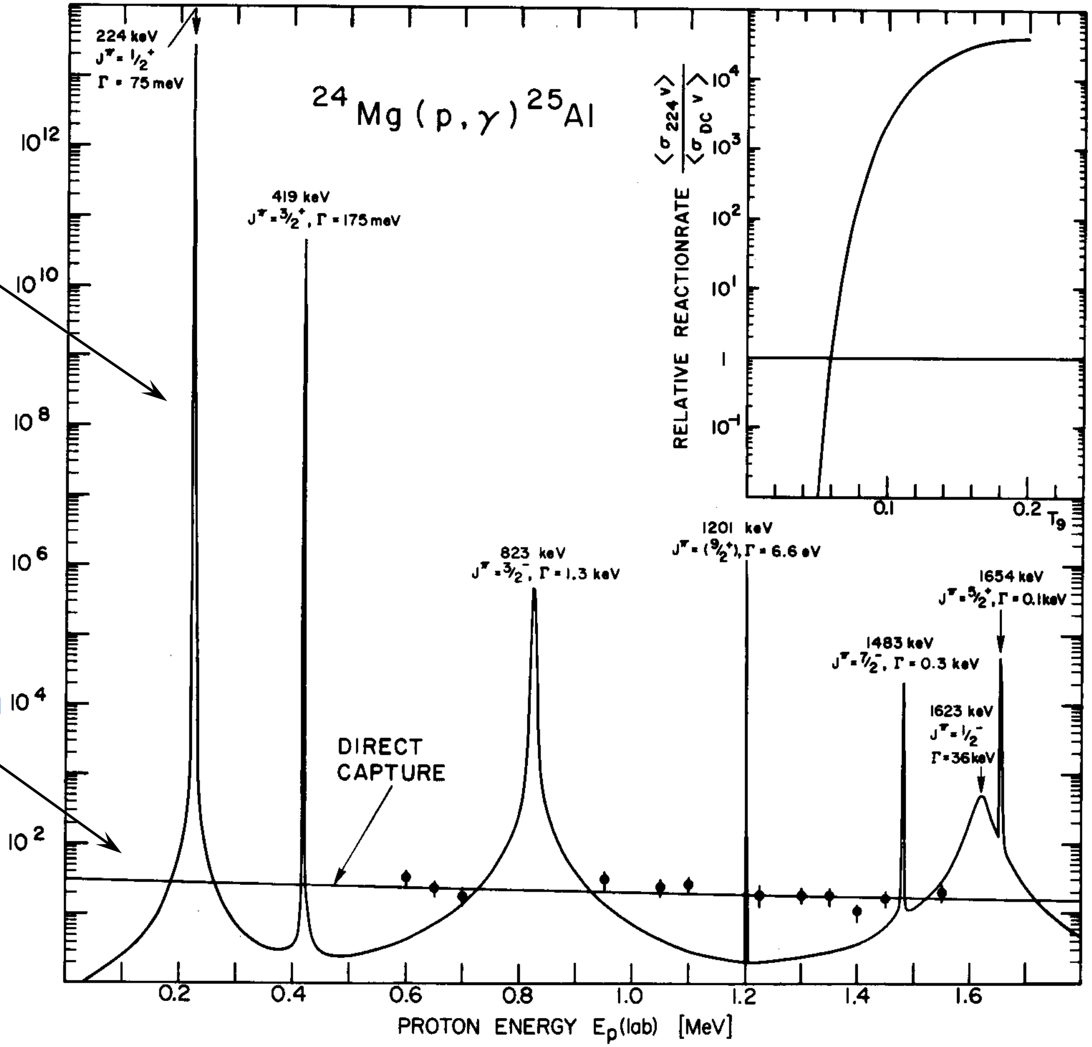
the cross section



… and the corresponding S-factor

non-constant S-factor
resonant contribution

almost constant S-factor
direct capture contribution



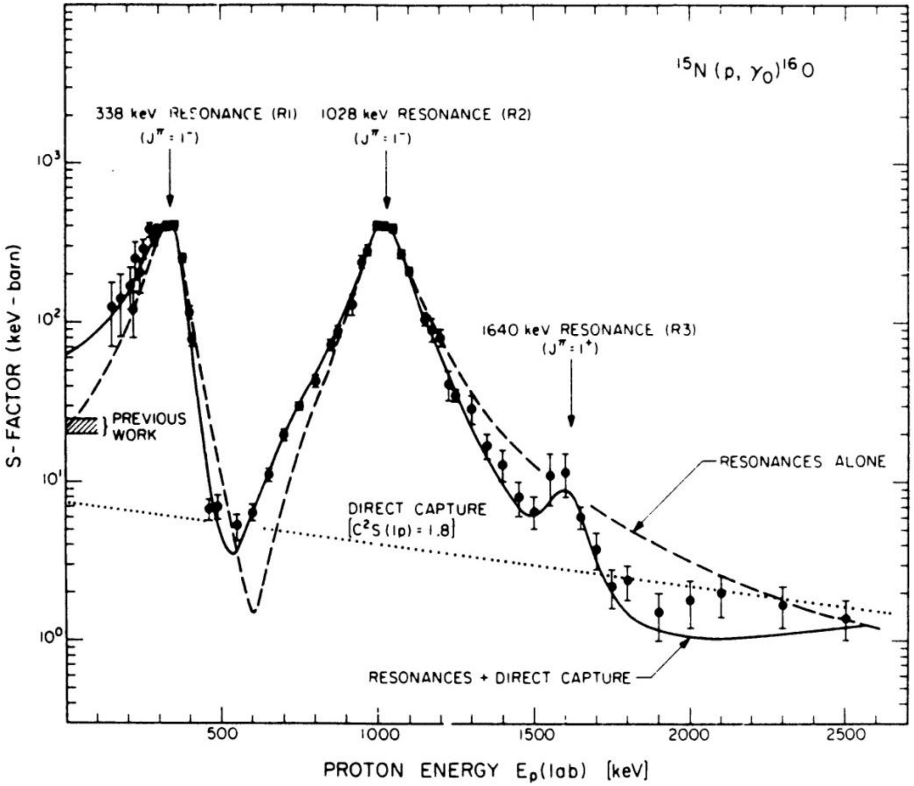
Note varying widths of resonant states !

reaction rate through: **broad resonances**

$\boxed{\Gamma \sim E_R}$

broader than the relevant energy window for the given temperature

resonances outside the energy range can also contribute through their wings



Breit-Wigner formula
+
energy dependence of partial and total widths

assume:

Γ_2= const, Γ = const and use simplified

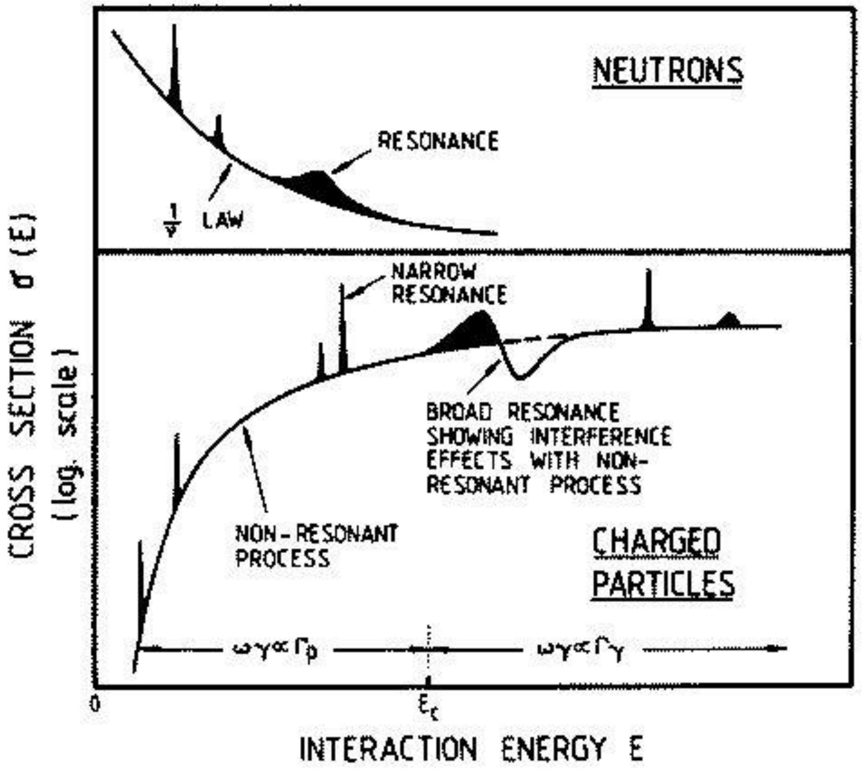
$$\boxed{\sigma(E) = \underbrace{\pi\bar{\lambda}^2\Gamma_1(E)\omega}_{\text{same energy dependence as in direct process}} \underbrace{\frac{\Gamma_2}{(E-E_R)^2+(\Gamma/2)^2}}_{\text{for } E \ll E_R \text{ very weak energy dependence}}}$$

N.B. overlapping broad resonances of same J^π → **interference effects** stellar reaction rate of nuclear reaction determined by the sum of contributions due to

- direct transitions to the various bound states
- all narrow resonances in the relevant energy window
- broad resonances (tails) e.g. from higher lying resonances
- any interference term

total rate

$$\boxed{\langle \sigma v \rangle = \sum_i \langle \sigma v \rangle_{\mathrm{DCi}} + \sum_i \langle \sigma v \rangle_{\mathrm{Ri}} + \langle \sigma v \rangle_{\mathrm{tails}} + \langle \sigma v \rangle_{\mathrm{int}\,erference}}$$



Rolfs & Rodney
Cauldrons in the Cosmos, 1988