Energy production in sun

- Sequence of nuclear reactions: proton-proton chain

1. $p + p \rightarrow {}^2D + e^+ + \nu_e$
2. ${}^2D + p \rightarrow {}^3He + \gamma$
3. ${}^3He + {}^3He \rightarrow {}^4He + p + p$

Total: $4\,p \rightarrow {}^4He + 2\,e^+ + 2\nu_e$

Introduction

proton proton chain

$4\mathrm{p} \rightarrow {}^4\mathrm{He} + 2\mathrm{e}^+ + 2\nu \quad \mathrm{Q} = 26.73\mathrm{MeV}$

possible first steps :

$\mathrm{p} + \mathrm{p} \rightarrow {}^2\mathrm{He}$ Diproton unbound nucleus

$({}^2\mathrm{He} + \mathrm{p} \rightarrow {}^3\mathrm{Li}$ also unbound)

- Weak interaction process

$\mathrm{p} + \mathrm{p} \rightarrow d + \mathrm{e}^+ + \nu$

Q - value: $\mathrm{Q} = 1.44\ \mathrm{MeV}$

kinetic energy: $\mathrm{E}_{\mathrm{kin}} = Q - 2m_e c^2 = 0.42 MeV$

$2m_p + 2m_e \rightarrow m_d + 3m_e \Rightarrow m_H + m_H \rightarrow m_{2H} + 2m_e$

kinetic energy is shared by positron and neutrino

- Theoretical cross section for the p + p reaction

total Hamiltonian : $\mathrm{H} = \mathrm{H}_\mathrm{n} + \mathrm{H}_\beta$

but H_β much smaller then $\mathrm{H}_\mathrm{n} \Rightarrow$ perturbation theory

Fermis Golden Rule :

$$\mathrm{d}\sigma = \frac{2\pi}{\hbar} \frac{\rho(E)}{v_i} \left| \langle f | \mathrm{H}_\beta | i \rangle \right|^2$$

$\rho(E)$ statistical factor, density of final states

v_i relative velocity in incident channel

$\left| \langle f | \mathrm{H}_\beta | i \rangle \right|$ transition matrix element

$|i\rangle \equiv \mathrm{p} + \mathrm{p}$

$|f\rangle \equiv d + \mathrm{e}^+ + \nu$

Phase space

$\rho(E) = \frac{dN}{dE}$ number of final states dN in energy interval dE (E and E + dE)

total phase space divided by unit volume $dn = V\frac{4\pi p^2 dp}{h^3}$

applying to pp reaction $\rightarrow$ states available for positrons and neutrinos

$$\mathrm{dN} = dn_e \cdot dn_\nu = \left(V\frac{4\pi p_e^2 dp_e}{h^3}\right)\left(V\frac{4\pi p_\nu^2 dp_\nu}{h^3}\right)$$

assume: - zero rest mass of neutrino $\mathrm{m}_\nu = 0$

- zero recoil energy of deuteron $\mathrm{E}_{kin,d} = 0$

$\Rightarrow$ Energy is shared by positron and neutrino

$$\mathrm{E} = \mathrm{E}_e + \mathrm{E}_\nu = \mathrm{E}_e + c\mathrm{p}_\nu$$

$p_\nu = \frac{E}{c} - \frac{E_e}{c}$ for constant p_e

$$\frac{dp_\nu}{dE} = \frac{1}{c} \qquad dp_\nu = \frac{1}{c}dE \qquad \frac{dp_\nu}{dE_e} = -\frac{1}{c}$$

neutrino level density: $\frac{dn_\nu}{dE} = \left(V\frac{4\pi p_\nu^2}{h^3}\frac{dp_\nu}{dE}\right) = V\frac{4\pi}{h^3}p_\nu^2\frac{dp_\nu}{dE} = V\frac{4\pi}{h^3}\frac{1}{c^2}(E - E_e)^2\frac{1}{c}$

$$dn_e = \left(V\frac{4\pi p_e^2 dp_e}{h^3}\right)$$

$$\rho(E) = \frac{dN}{dE} = dn_e \cdot \frac{dn_\nu}{dE} = \left(V\frac{4\pi p_e^2 dp_e}{h^3}\right)\cdot V\frac{4\pi}{h^3}\frac{1}{c^2}(E - E_e)^2\frac{1}{c}$$

$$\rho(E) = V^2\frac{16\pi^2}{h^6}\frac{1}{c^3}p_e^2(E - E_e)^2 dp_e = \rho(E_e)dp_e$$

Matrix element

Back to the cross section for the p + p reaction, cross section depends only on positron variables

Fermis Golden Rule:

$$d\sigma = \frac{2\pi}{\hbar}\frac{\rho(E_e)}{v_i}\left|\langle f\,|H_\beta|\,i\rangle\right|^2 dp_e$$

Matrix element: $H_{if} = \langle f\,|H_\beta|\,i\rangle = \int_{Vol}\psi_f^* H_\beta \psi_i\, d\tau = \int_{Vol}\left[\psi_d\psi_e\psi_\nu\right]^* H_\beta\psi_i\, d\tau$

Wave function of positron and neutrino: plane waves

$$\psi_e = \frac{1}{V^{1/2}}\exp(i\vec{k}_e\vec{r}) \quad \text{and} \quad \psi_\nu = \frac{1}{V^{1/2}}\exp(i\vec{k}_\nu\vec{r})$$

Leptons are normalized to the volume.

For initial protons: $\int_{Vol}\psi_i^*\psi_i\, d\tau = 1$

Deuteron wave function vanishes rapidly outside nuclear radius $r_d = 1.7$ fm.

Mean neutrino energy from phase space: $\overline{E}_\nu = 0.26$ MeV.

$\Rightarrow\ k_\nu r_d = 2.2\times10^{-3}$

expansion of wave functions: $\psi_e = \frac{1}{V^{1/2}}\left[1 + i(\vec{k}_e\vec{r}) + \ldots\right]$

$$\psi_\nu = \frac{1}{V^{1/2}}\left[1 + i(\vec{k}_\nu\vec{r}) + \ldots\right]$$

Cross section

Matrix element after expansion :

$$H_{if} = \frac{1}{V}\int_{Vol} \Psi_d^* \mathrm{H}_\beta \Psi_i \, d\tau$$ strength of weak interaction H_β is governed by coupling constant g

(analog to electric charge as coupling constant α in EM interaction)

$$H_{if} = \frac{1}{V} g \int_{Vol} \Psi_d^* \Psi_i \, d\tau = \frac{1}{V} g M_{space} M_{spin}$$

$$d\sigma = \frac{2\pi}{\hbar}\frac{\rho(E_e)}{v_i}\left|\langle f | \mathrm{H}_\beta | i \rangle\right|^2 dp_e = \frac{2\pi}{\hbar}\frac{1}{v_i} V^2 \frac{16\pi^2}{h^6}\frac{1}{c^3} p_e^2 (E-E_e)^2 \left(\frac{1}{V} g M_{space} M_{spin}\right)^2 dp_e$$

$$\sigma = \frac{2\pi}{\hbar}\frac{1}{v_i}\frac{16\pi^2}{h^6}\frac{1}{c^3} g^2 M_{space}^2 M_{spin}^2 \int_0^E p_e^2 (E-E_e)^2 dp_e$$

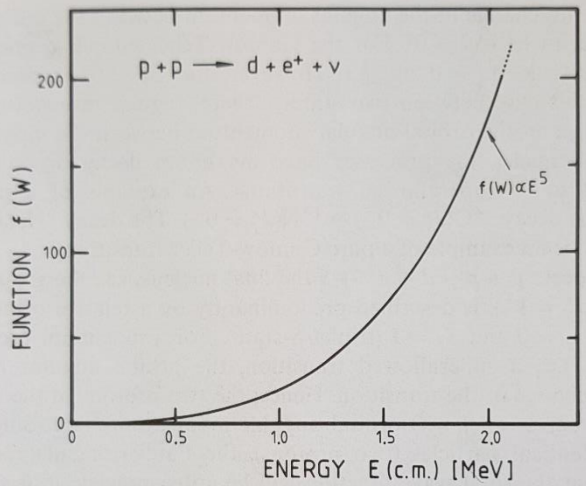
new variable : $w = \frac{E + m_e c^2}{m_e c^2}$

substitution : $\int_0^E p_e^2 (E-E_e)^2 dp_e = \frac{(m_e c^2)^5}{c^3}\int_1^w (w_e^2 - 1)^{1/2} (w - w_e)^2 w_e dw_e = \frac{(m_e c^2)^5}{c^3} f(w)$

integral is proportional to E^5 $f(w) \propto w^5 \propto E^5$

$$\sigma = \frac{1}{v_i}\left[\frac{2\pi}{\hbar}\frac{16\pi^2}{h^6}\frac{1}{c^3}\frac{(m_e c^2)^5}{c^3}\right] f(w) g^2 M_{space}^2 M_{spin}^2$$ constant values $$\alpha = \left[\frac{2\pi}{\hbar}\frac{16\pi^2}{h^6}\frac{1}{c^3}\frac{(m_e c^2)^5}{c^3}\right] = 1.45 \cdot 10^{70}\frac{1}{eV^2 \cdot s \cdot cm^6}$$

$$\sigma = \alpha \frac{1}{v_i} f(w) g^2 M_{space}^2 M_{spin}^2$$



The function $f(W)$, which is a measure of the total phase space available for t $e^+ + \nu$ reaction, is shown as a function of energy E. As might be expected, the ava pace increases rapidly with E. For large values of energy, the phase space and hen ty for the process increase as the fifth power of the energy.

illustration of this function $f(W)$ is presented in Figure 6.3. F s of W, i.e., large energies E, the function $f(W)$ approaches the for $^5 \propto E^5$.

al cross section is given now by the equation

$$1 \quad \frac{2\pi}{\hbar} \frac{16\pi^2}{c^3 h^6} \frac{(m_e c^2)^5}{c^3} f(W) g^2 M_{\text{spin}}^2 M_{\text{space}}^2$$

$$\propto \frac{1}{v_i} f(W) g^2 M_{\text{spin}}^2 M_{\text{space}}^2 , \qquad (6.$$

epresents the constants, which, when replaced by their numeric ult in

$$\frac{m^5 c^4}{2\pi^3 \hbar^7} = 1.45 \times 10^{70} \text{ eV}^{-2} \text{ s}^{-1} \text{ cm}^{-6} .$$

ations of allowed weak-interaction phenomena indicate (Bla6 t among all possible types of couplin

Weak interaction couplings

(i) Spins of leptons in final state are ↑↓ antiparallel

$s_e + s_\nu = s_{tot} = 0$

Fermi or vector coupling $g=C_V$

change in angular momentum: $\Delta J=0$

e.g. Fermi transition: $^{14}O(J^\pi_i=0^+) \rightarrow {}^{14}N(J^\pi_f=0^+)$

(ii) Spins of leptons in final state are ↑↑ parallel

$s_e + s_\nu = s_{tot} = 1$

Gamow-Teller or axial-vector coupling $g=C_A$

change in angular momentum: $\Delta J=0, \pm 1$

However, $\Delta J=0$ not possible between two states of zero angular momentum.

e.g. GT transition: $^{6}He(J^\pi_i=0^+) \rightarrow {}^{6}Li(J^\pi_f=1^+)$

Implications for pp reaction

$p + p \rightarrow d + e^+ + \nu$

(i) Spins of ground state of deuteron, final state ^2H ($J^\pi_f = 1^+$) described by l_f=0 and s_f=1, triplet s-state, only small d-state admixture.

(ii) Max. probability to react is related to process without change in angular momentum. Two protons are in l_i=0, implies a symmetric wave function for Ψ_{space} .

(iii) Two identical Fermions (2 protons) → Pauli principle requires singulett configuration s_i = 0 (↑↓) to get an antisymmetric wave function. ($\Psi_{tot} = \Psi_{space} \Psi_{spin}$).

(iv) $s_i=0, l_i=0 \rightarrow s_f=1, l_f=0$
Gamow-Teller-Transition with coupling constant $g=C_A$

(v) C_A is determined from lifetimes of GT transitions in neutron, triton, ^6He decay.

Final results for pp reaction

$p + p \rightarrow d + e^+ + \nu$

(vi) M_{spin}

summing over final states, averaging over initial states, division by 2 due to the fact of identical particles

$$M_{spin} = 3/2$$

(vii) M_{space}

numerical integration of product of radial wave functions of protons and deuteron.

$$M_{space} = \int_0^\infty \chi_f(r)\chi_i(r)r^2 dr$$

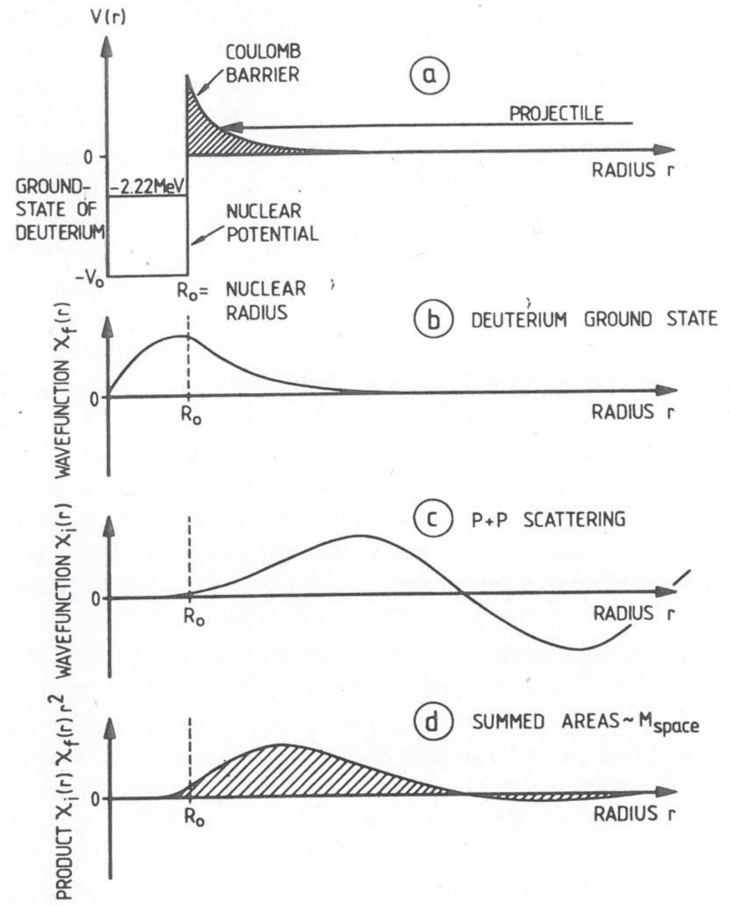


FIGURE 6.4. Shown schematically are a few ingredients used in the numerical evaluation of the space matrix element M_{space} for the $p + p \rightarrow d + e^+ + \nu$ reaction. The potential is shown in (*a*), where, for a given nuclear radius R_0, the observed binding energy of the deuterium determines the potential depth V_0. The deuterium radial wave function $\chi_f(r)$ is determined by the potential $V(r)$. Because of the loosely bound ground state, $\chi_f(r)$ extends far outside R_0 with appreciable amplitudes (*b*). The initial wave function $\chi_i(r)$ is obtained from $p + p$ elastic scattering data, which gives (*c*) a small amplitude for $r \leq R_0$ and has the usual oscillating pattern of a plane wave for $r \gg R_0$. The radial integrand in M_{space} (*d*) then has its major contributions in regions far outside R_0 (*hatched areas*).

Final results for pp reaction

$p + p \rightarrow d + e^+ + \nu$

(vi) M_{spin} summing over the final states, averaging over the initial states, dividing by 2 due to the fact of identical particles $M_{spin} = 3/2$.

(vii) M_{space} numerical integration of product of radial wave functions of protons and deuteron.

$$M_{space} = \int_0^\infty \chi_f(r)\chi_i(r) r^2 dr$$

Final result: Total cross section at E_p=1MeV

$$\sigma(E) \sim 10^{-47}\ \mathrm{cm}^2$$

non resonant reaction, smooth variation with energy at T_6=15

$$\langle \sigma v \rangle_{pp} = 1.19 \times 10^{-43} \left[cm^3 \frac{1}{s} \right]$$

What is happening inside the sun?

$p + p \rightarrow d + e^+ + \nu$

Sun interior is a mixture of H and He

Assumption: $X_H \sim X_{He} \sim 0.5$ density $\rho \sim 100$ g/cm^3

Mean lifetime of proton against combustion

$$\tau_H = \frac{1}{N_H \langle \sigma v \rangle_{pp}} = 0.9 \cdot 10^{10}\, year$$

Overall rate of the conversion of four protons into ^{4}He is governed by the pp reaction!

$\tau \sim 10^{10}$ years ~ age of oldest known stars!

What is happening inside the sun next?

$p + p \rightarrow d + e^+ + \nu$

Burning of deuteron:

$p + d \rightarrow {}^3He + \gamma$ Q-value: 5.494 MeV, EM interaction

$$\frac{dd}{dt} = r_{pp} - r_{pd} \qquad r_{12} = \frac{N_1 N_2}{1+\delta_{12}} \langle \sigma v \rangle_{12}$$

$$\text{quasi equilibrium}: \frac{dd}{dt} = 0$$

$$\left(\frac{d}{p}\right) = \frac{\langle \sigma v \rangle_{pp}}{2\langle \sigma v \rangle_{pd}} \qquad \left(\frac{d}{p}\right) = 5.6 \times 10^{-18}$$ at $T_6=5$

Mean lifetime of deuteron against combustion $\tau(D)=1.6$ s !

Reaction rates and network

$$\left(\frac{dN_1}{dt}\right)_2 = -(1+\delta_{12})\, r = -N_1 N_2 \langle\sigma v\rangle$$

$$= -N_1 \rho N_A \frac{X_2}{A_2}\langle\sigma v\rangle = -N_1 \rho N_A Y_2 \langle\sigma v\rangle$$

$$r = \frac{1}{1+\delta_{12}} N_1 N_2 \langle\sigma v\rangle$$

$$N_i = \rho N_A \frac{X_i}{A_i} = \rho N_A Y_i$$

Number density N
Matter density ρ
Avogadro's constant N_A
Mass fraction X
Atomic mass A
Mol fraction Y

The solar ppI chain consists of the following reactions:
$p(p,e^+\nu)d$ (11), $d(p,\gamma)^3He$ (12), $^3He(^3He,2p)^4He$ (33)
with $Y_1 = Y_p,\ Y_2 = Y_d,\ Y_3 = Y_{^3He},\ Y_4 = Y_{^4He}$

$$\begin{aligned}
\dot{Y}_1 &= \rho N_A(-\langle\sigma v\rangle_{11} Y_1^2 - \langle\sigma v\rangle_{12} Y_1 Y_2 + \langle\sigma v\rangle_{33} Y_3^2) \\
\dot{Y}_2 &= \rho N_A(+\frac{1}{2}\langle\sigma v\rangle_{11} Y_1^2 - \langle\sigma v\rangle_{12} Y_1 Y_2) \\
\dot{Y}_3 &= \rho N_A(+\langle\sigma v\rangle_{12} Y_1 Y_2 - \langle\sigma v\rangle_{33} Y_3^2) \\
\dot{Y}_4 &= \rho N_A(+\frac{1}{2}\langle\sigma v\rangle_{33} Y_3^2)
\end{aligned}$$

The factors 1/2 avoid double-counting of identical nuclei.

proton-proton chain

$p + p \rightarrow d + e^+ + \nu$

$p + d \rightarrow {}^3He + \gamma$

86% | 14%

$^3He + {}^3He \rightarrow {}^4He + 2p$

PP-I

$Q_{eff} = 26.20$ MeV

$^3He + {}^4He \rightarrow {}^7Be + \gamma$

99.7% | 0.3%

$^7Be + e^- \rightarrow {}^7Li + \nu$

$^7Li + p \rightarrow 2\ {}^4He$

PP-II

$Q_{eff} = 25.66$ MeV

$^7Be + p \rightarrow {}^8B + \gamma$

$^8B \rightarrow {}^8Be + e^+ + \nu$

$\downarrow$

$2\ {}^4He$

PP-III

$Q_{eff} = 19.17$ MeV

net result: $4p \rightarrow {}^4He + 2e^+ + 2\nu + Q_{eff}$