Last lecture: Comparison of interactions

Bremsstrahlung loss and ionisation loss for electrons and protons in copper

electron 2-4 MeV

crit. energy 24 MeV

proton > 3 GeV
Last lecture: Bremsstrahlung

\[ d\sigma = 4Z^2r_e^2\alpha \frac{d\nu}{\nu} \left[ (1 + \varepsilon^2)\left(\frac{\phi_1(\xi)}{4} - \frac{1}{3}\ln Z - f(Z)\right) - \frac{2}{3}\varepsilon\left(\frac{\phi_2(\xi)}{4} - \frac{1}{3}\ln Z - f(Z)\right) \right] \]

Differential Bremsstrahlung cross section

\( \varepsilon = E/E_0 \)

\( E_0 \) : incoming initial electron energy

\( E \) : electron energy after radiation

\( h\nu \) : photon energy

Important screening Parameter : \( \xi = \frac{100m_e c^2 h\nu}{E_0 E Z^{1/3}} = \frac{100m_e c^2 h\nu}{E_0 Z^{1/3} E} \)

Screening depends on

(i) initial energy \( \xi \sim 12-25 \) for 1 MeV electrons, no screening

\( \xi \sim 0.1-0.3 \) for 100 MeV electrons, complete screening

(ii) energy of Bremsstrahlung photon vs. remaining energy varies from small numbers for soft photons to highest numbers for head on collisions causing high energy radiation
Two solutions for limiting extrem cases:

For $\xi \gg 1$ - no screening:

$$d\sigma = 4Z^2 r_e^2 \alpha \frac{d\nu}{\nu} \left[ (1 + \varepsilon^2 - \frac{2\varepsilon}{3}) \left( \ln \frac{2E_0E}{m_e c^2 \hbar \nu} - \frac{1}{2} - f(Z) \right) \right]$$

For $\xi \approx 0$ - complete screening:

$$d\sigma = 4Z^2 r_e^2 \alpha \frac{d\nu}{\nu} \left[ (1 + \varepsilon^2 - \frac{2\varepsilon}{3}) \left( \ln(183Z^{-1/3}) - f(Z) \right) + \frac{\varepsilon}{9} \right]$$

Radiative energy loss from integral (xsection $\times$ photon energy) $\times$ N

$$- \left( \frac{dE}{dx} \right)_{rad} = N \int_0^{\nu_0} h\nu \frac{d\sigma}{d\nu} (E_0, \nu) d\nu = NE_0 \phi_{rad}$$

$$N = \frac{\rho N_a}{A} \quad \text{and} \quad \nu_0 = \frac{E_0}{h}$$

$$\phi_{rad} = \frac{1}{E_0} \int_0^{\nu_0} h\nu \frac{d\sigma}{d\nu} (E_0, \nu) d\nu$$

$N$: number of atoms/cm$^3$
Last lecture: Bremsstrahlung

For $m_e c^2 \ll \xi \ll 137 m_e c^2 Z^{-1/3}$, $\xi \gg 1$ no screening:

$$\phi_{rad} = 4Z^2 r_e^2 \alpha \left( \ln \frac{2E_0}{m_e c^2} - \frac{1}{3} - f(Z) \right)$$

For $E_0 \gg 137 m_e c^2 Z^{-1/3}$, $\xi \approx 0$ complete screening:

$$\phi_{rad} = 4Z^2 r_e^2 \alpha \left( \ln(183Z^{-1/3}) + \frac{1}{18} - f(Z) \right)$$

High energy limit is independent of electron energy $E_0$

$$\phi_{rad} = 4Z^2 r_e^2 \alpha \left( \ln(183Z^{-1/3}) + \frac{1}{18} - f(Z) \right)$$
Last lecture: Collisions vs. Bremsstrahlung

\[-\left(\frac{dE}{dx}\right)_{\text{coll}} = 2\pi N_a r_e^2 m_e c^2 \rho \frac{Z}{A} \frac{1}{\beta^2} \left[ \ln\left(\frac{\tau^2 (\tau + 2)}{2 (I / m_e c^2)^2}\right) + F(\tau) - \delta - 2 \frac{C}{Z} \right] \]

Radiative energy loss from integral (xsection × photon energy) × N

\[-\left(\frac{dE}{dx}\right)_{\text{rad}} = N \int_0^{\nu_0} h \nu \frac{d\sigma}{d\nu} (E_0, \nu) d\nu = NE_0 \phi_{\text{rad}} \]

\[\phi_{\text{rad}} = \frac{1}{E_0} \int_0^{\nu_0} h \nu \frac{d\sigma}{d\nu} (E_0, \nu) d\nu = 4Z^2 r_e^2 \alpha \left( \ln(183Z^{-1/3}) + \frac{1}{18} - f(Z) \right) \]

Compare collision or ionisation vs Bremsstrahlung:
- ionisation increases logarithmic in \( E \) and linear in \( Z \)
- bremsstrahlung increases linear in \( E \) and quadratic in \( Z \)
- ionisation: quasi-continuous energy loss
- bremsstrahlung: emission of one or two photons can give away full electron energy -> huge fluctuations
Critical energy $E_c$:

$$\left( \frac{dE}{dx} \right)_{rad} = \left( \frac{dE}{dx} \right)_{coll} \quad \text{for} \quad E = E_c \quad E_c \approx \frac{800\text{MeV}}{Z + 1.2}$$

Typical values:

<table>
<thead>
<tr>
<th>material</th>
<th>critical energy $E_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pb</td>
<td>9.51 MeV</td>
</tr>
<tr>
<td>Al</td>
<td>51.0 MeV</td>
</tr>
<tr>
<td>Fe</td>
<td>27.4 MeV</td>
</tr>
<tr>
<td>Cu</td>
<td>24.8 MeV</td>
</tr>
<tr>
<td>Luft</td>
<td>102 MeV</td>
</tr>
<tr>
<td>NaI</td>
<td>17.4 MeV</td>
</tr>
<tr>
<td>H2O</td>
<td>92 MeV</td>
</tr>
</tbody>
</table>
**Radiation length**

*distance in matter related to a radiative energy loss by 1/e*

\[-dE/E = N\phi_{rad} \, dx\]

high energetic limit (\(\xi \approx 0\)):

(i) coll. loss ignored

(ii) \(\phi_{rad}\) is independent from energy

Solution of differential equation

\[E = E_0 \exp\left(-\frac{x}{L_{rad}}\right)\]

\(x\) is distance in matter and \(L_{rad} = 1/N\phi_{rad}\) is radiation length

\[
\frac{1}{L_{rad}} \approx \left[4Z(Z+1)\frac{\rho N_a}{A}\right]r_e^2\alpha\left[\ln\left(183Z^{-1/3}\right) - f(Z)\right]
\]

**Typical values:**

<table>
<thead>
<tr>
<th>Material</th>
<th>cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pb</td>
<td>0.56</td>
</tr>
<tr>
<td>Al</td>
<td>8.90</td>
</tr>
<tr>
<td>Fe</td>
<td>1.76</td>
</tr>
<tr>
<td>Cu</td>
<td>1.43</td>
</tr>
<tr>
<td>NaI</td>
<td>2.59</td>
</tr>
<tr>
<td>BGO</td>
<td>1.12</td>
</tr>
<tr>
<td>Org.Scint.</td>
<td>42.40</td>
</tr>
<tr>
<td>(H_2O)</td>
<td>36.10</td>
</tr>
<tr>
<td>Air</td>
<td>30050.00</td>
</tr>
</tbody>
</table>
Summary: interactions of electrons

\[
\left( \frac{dE}{dx} \right)_{\text{tot}} = \left( \frac{dE}{dx} \right)_{\text{rad}} + \left( \frac{dE}{dx} \right)_{\text{coll}}
\]

Energy loss via collisions with electrons:

\[
-\left( \frac{dE}{dx} \right)_{\text{coll}} = 2\pi N_a r_e^2 m_e c^2 \rho \frac{Z}{A} \frac{1}{\beta^2} \left[ \ln \left( \frac{\tau^2(\tau + 2)}{2(I/m_e c^2)^2} \right) + F(\tau) - \delta - 2 \frac{C}{Z} \right]
\]

\( \tau \): Kin. energy in units of \( m_e c^2 \)

\( F(\tau) \): \( F(\tau) = 1 - \beta^2 + \ldots \) electron \quad \( F(\tau) = 2 \ln 2 - \frac{\beta^2}{12} (23 + \ldots) \) positron

Kritical energy \( E_c \):

\[
\left( \frac{dE}{dx} \right)_{\text{rad}} = \left( \frac{dE}{dx} \right)_{\text{coll}} \quad \text{for } E = E_c \quad E_c \approx \frac{800\text{MeV}}{Z + 1.2}
\]
Three regions:

1. **Single scattering:** very thin absorber, probability of more than one scattering very small, Rutherford formula, target like

2. **Plural scattering,** scatterings $N<20$, complicated calculation, neither Rutherford nor statistical methods.

3. **Multiple scattering:** many independent scatterings $N>20$, small energy loss -> probability distribution for the net angle of deflection as function of thickness

\[ \frac{d\sigma}{d\Omega} = z_2^2 z_1^2 r_e^2 \frac{(m_e c / \beta p)^2}{4 \sin^4(\theta / 2)} \]
Angular distribution of $E=15.7$ MeV electrons scattered from thin Au foil

compare:
- experimental data
- Gaussian approximation
Range and energy loss

Variation in range and energy loss caused by statistics of absorbing collisions.

Energy loss distribution
- average energy loss – $(dE/dx)$
- energy loss straggling
- range straggling

Different but related question:
Fluctuation in energy loss – $(dE/dx)$ for a fixed thickness of absorber?

Heavy charged particle
electron

Absorption of homogeneous in aluminium β-rays
Thick absorbers: energy loss

thick absorber:  
- N number of collisions is large, \( N \to \infty \)
- small individual energy loss in absorber
- sum of many independent losses causes Gaussian form

\[
f(x, \Delta) \propto \exp\left(\frac{-(\Delta - \bar{\Delta})^2}{2\sigma^2}\right)
\]

- \( x \): absorber thickness
- \( \Delta \): energy loss in absorber
- \( \bar{\Delta} \): mean energy loss
- \( \sigma \): standard deviation

non relativistic heavy particles (Bohr):

\[
\sigma^2 = 4\pi N_a r_e^2 (m_e c^2)^2 \rho \frac{Z}{A} x = 0.1569 \rho \frac{Z}{A} x \text{[MeV}^2\text{]}.
\]
distinguishing parameter: \( \kappa = \Delta / W_{\text{max}} \)

mean E-loss / max E-transfer

very thin absorber material with \( Z, A \)

first term of Bethe-Bloch:

\[
\Delta \cong \xi = 2\pi N_a m_e r_e^2 c^2 \rho \left( \frac{Z}{\beta} \right)^2 \frac{Z}{A} \cdot x \quad (\text{mean energy loss})
\]

example for \( z = 1 \):

\[
\xi = \frac{0.1536 Z}{\beta^2} \frac{Z}{A} \cdot x \quad [\text{keV}] \quad x \text{ in mg/cm}^2
\]

1 cm thickness Ar gas and \( \beta = 1 \) \( \xi = 0.123 \text{ keV} \)
Landau theory: very thin absorbers

Probability distribution for energy loss $\Delta$ of very thin absorber with thickness $x$ (with $\kappa \leq 0.01$, thin absorber, $\beta \approx 1$) Landau's solution:

$$f(x, \Delta) = \frac{1}{\xi} \cdot \frac{1}{\pi} \int_0^\infty e^{-\ln u - \lambda u} \sin(\pi u) du$$

$\lambda$ is normalized deviation from most probable energy loss $\Delta^{m.p.}$

$$\lambda = \frac{\Delta - \Delta^{m.p.}}{\xi}$$

most probable energy loss: $\Delta^{m.p.} = \xi \left\{ \ln \frac{2mc^2\beta^2\gamma^2\xi}{I^2} - \beta^2 + 1 - \gamma_E \right\}$ and $\gamma_E = 0.577$

Approximation:

$$f(x, \Delta) = \frac{1}{\xi} \cdot \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} (\lambda + e^{-\lambda}) \right)$$

![Graph showing the probability distribution](image)
Energy loss of electrons vs Landau distribution
Energy loss of pions in silicon

\[ \Delta/x \quad (\text{MeV g}^{-1} \text{ cm}^2) \]

\[ f(\Delta/x) \]

500 MeV pion in silicon

- 640 \( \mu \text{m} \) (149 mg/cm\(^2\))
- 320 \( \mu \text{m} \) (74.7 mg/cm\(^2\))
- 160 \( \mu \text{m} \) (37.4 mg/cm\(^2\))
- 80 \( \mu \text{m} \) (18.7 mg/cm\(^2\))

\[ \Delta_{p}/x \]

\[ \text{Mean energy loss rate} \]

\[ w \]
Compare collision or ionisation vs Bremsstrahlung:
- Ionisation increases logarithmic in E and linear in Z
- Bremsstrahlung increases linear in E and quadratic in Z
- Ionisation: quasi-continuos energy loss
- Bremsstrahlung: emission of one or two photons can give away full electron energy -> huge fluctuations