Multipole Decomposition of Residual Interactions

We have seen that the relative energies of 2-particle systems affected by a residual interaction depend SOLELY on the angles between the two angular momentum vectors, not on the radial properties of the interaction (which just give the scale).

We learn a lot by expanding the angular part of the residual interaction,

\[ H_{\text{residual}} = V(\theta, \phi) \]

in spherical harmonics or Legendre polynomials.
Multipole Decomposition of Residual Interactions

Can expand an interaction ($\Theta$) in series using complete set of functions $P_k (\cos \Theta)$

$$\nu \left( \vec{r}_{12} \right) \text{ is 2-body interaction. Write:}$$

$$\nu \left( \vec{r}_{12} \right) = \sum_k \nu_k P_k (\cos \Theta)$$
Now, we re-consider the energy shifts

$$\Delta E \left( j_1, j_2, J \right) = \sum_k F^k A_k$$

where $$F^r_k = \int R_{n_1}^2 R_{n_2}^2 \nu_k \, dr_1 \, dr_2$$

and $$A = \frac{(-1)^{j_1+j_2+j}}{2k+1} \frac{4\pi}{\left\langle l_1, j_1 \parallel Y_k \parallel l_1, j_1 \right\rangle} \left\langle l_2, j_2 \parallel Y_k \parallel l_2, j_2 \right\rangle \alpha \left\langle j_1, j_2, J \right\rangle \left\langle j_1, j_1, k \right\rangle$$

**Limitation on k**

**Key to physics**

**Relation of "probe" to force!!**

Triangle conditions: $Y_k$ carries ang. mom. $k$

:: $l_1, l_2, k$ must "close"

(Rigorously, applies to direct terms in interactions, not to exchange terms)

$k < \min \left( 2l_1, 2l_2, 2j_1, 2j_2 \right)$
$k$ Restriction on Multipoles

Reflects important physical idea

**Effects** – great simplification

e.g., \( d_{3/2}^2 \) \hspace{1cm} \( k_{\text{max}} = 3 \)

since \( k = \text{even} \) \( \rightarrow \) \( k = 0, 2 \) only

\( d_{3/2} d_{5/2} \) \hspace{1cm} \( k = 0, 2 \) only

\( d_{3/2} g_{9/2} \) \hspace{1cm} \( k = 0, 2 \) only

\( d_{3/2} s_{1/2} \) \hspace{1cm} \( k = 0 \) only !

\( s_{1/2}^2 \) \hspace{1cm} \( k = 0 \) only !

\( g_{7/2}^2 \) \hspace{1cm} \( k = 0, 2, 4, 6 \)

Physical idea

Relation of probe to "probe"

2 particles in \( s_{1/2} \) orbit are insensitive to multipoles above monopole (i.e., constant) force ! ! What means? Why?
Configuration for which only a monopole and/or quadrupole force applies

Monopole only

\[ s^{2}_{1/2}, \quad p^{2}_{1/2} \]

\[ s_{1/2} + \text{anything} \]

Monopole + Quadrupole

\[ d^{2}_{3/2} \]

\[ p^{2}_{3/2} \]

\[ d_{3/2} + \text{anything} \]

\[ d_{3/2} + \text{anything} \]

\[ k = 0 + 2 + 4 \]

\[ d^{2}_{5/2} \]

\[ f^{2}_{5/2} \]

\[ d_{5/2} + \text{anything} \]

\[ f_{5/2} + \text{anything} \]

All forces identical!

All forces effectively equivalent to a quadrupole force.

\[ \delta \sim Q \]
Probes and “probees”
Probes and Objects Probed

Relative size of probe and object studied must be similar.

(Explains need for high energy accelerators for particle physics)

In our case, the probe (of the force) is the nucleon in an orbit, in particular the angular “size” (spread) of the wave function in a given magnetic substrate, \( m \), which is directly related to \( j \).

\[
\begin{align*}
\text{low } j & \quad +j \\
\quad & \quad -j \\
\text{high } j & \quad +j \\
\quad & \quad -j
\end{align*}
\]

s state: \( \Psi \) very spread out in space

\[
\therefore \text{ Can’t sense details of a force. Particles “always” in contact. A } \delta \text{ force appears the same as a constant force ! !}
\]

High \( j \) - sensitive to details of force—each \( m \) value highly localized.
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$$\nu \left( \vec{r}_{12} \right) = \sum_k\nu_k P_k (\cos \Theta)$$

Multiply on both sides by $P_{kH} (\cos \Theta)$ and integrate

$$\int \nu \left( \vec{r}_{12} \right) P_{k'} (\cos \Theta) d\Theta = \int \sum_k \nu_k P_k (\cos \Theta) P_{k'} (\cos \Theta) d\Theta$$

Use $Y_{k0} (\Theta) = \frac{2k+1}{4\pi} P_k \cos \Theta$

$$\int \nu \left( \vec{r}_{12} \right) P_{k'} (\cos \Theta) d\Theta = \sqrt{\frac{4\pi}{2k+1}} \sqrt{\frac{4\pi}{2k'+1}} \int \nu_k Y_{k0} (\Theta) Y_{k'0} (\Theta) d\Theta$$

Use $\int Y_{k m} (\Theta) d\Theta Y_{k'm'} (\Theta)\sin \Theta d\Theta d\Phi = \delta_{k k'} \delta_{m m'}$

$$\int \nu \left( \vec{r}_{12} \right) P_{k'} (\cos \Theta) d\Theta = \frac{4\pi}{2k+1} \nu_k \frac{1}{2\pi}$$

$$\nu_k = \frac{2k+1}{2} \int \nu \left( \vec{r}_{12} \right) P_k (\cos \Theta) d\Theta$$

Expansion coefficient

Now, use this in the expression for the energy shifts in configuration $|j_1 j_2 J \rangle$ in presence of a residual interaction.
Apply multipole expansion to $\delta$ force

For derivation, see Heyde

Result:

$$V_k (r_1, r_2) = \frac{2k+1}{4\pi} \frac{\delta(r_1 - r_2)}{r_1 r_2}$$

High $k$ multipoles most important.

$k = 0$ component: monopole – overall shift of all $J$ levels in $| j_1, j_2, J \rangle$

$k = 2$ same as quad. force

$$\left( d^{2}_{3/2} \right)$$  $k = 0, 2$ only

Often difficult to distinguish forces from data on low $j$ orbits.
Effects of Different Multipoles

Look at structure of:

\[ |\Psi_{12}(\Theta_{12})|^2 \] and of \( P_k(\cos \Theta) \)

Each 2-particle configuration, \( |j_1 j_2 J\rangle \), will have \( \Psi_{\text{max}}(\Theta_{12}) \) at the semi-classical angle and will fall off rapidly away from this angle.

\[ \text{e.g.,} \]

\[ \hbar_{11/2}^2 \quad J = 0 \quad \Psi_{\text{max}} \sim 180^\circ \]
\[ \quad J = 2 \quad \Psi_{\text{max}} \sim 155^\circ \]
\[ \quad J = 4 \quad \Psi_{\text{max}} \sim 134^\circ \]
\[ \quad J = 6 \quad \Psi_{\text{max}} \sim 112^\circ \]
\[ \quad J = 8 \quad \Psi_{\text{max}} \sim 84^\circ \]

If \( P_k(\cos \Theta) \) is large at one of those angles, an attractive force will lower the energy of that level.

Reason: energy lowering \( \sim \int \Psi^k P_k \Psi \delta \Theta \)

Integrand is only large when both \( \Psi(\Theta), P_k(\Theta) \) are large.
Semi-classical angles for the configurations $|j, j_i, J>$

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So — recall $V = -V_0 \delta (R_{12})$, so $P_k \rightarrow -P_k$: positive values are attractive

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<th>$k$</th>
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<td>10+</td>
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Note: 0+ always attractive — each $k$ lowers more $k$'s allowed, more lowered higher $j$

$\therefore$ 0+ for higher $j$'s ($\therefore$ higher $k$'s allowed) lowered more
Multiplet Splittings for $\delta$ force in $(h_{11/2})^2$ configuration as function of multipoles included

$k = 0 + 2$  
parabolic

All $k$  
$\delta$ fct. results seen before

Separation of $0^+$ from other states is specifically an effect of higher order terms—i.e., shorter range terms. Pauli effect appears in highest orders!

Note: all multipoles affect $0^+$ since $P_k (\cos 180^\circ) = 1$ 
For other spins cancellations occur. 
$P_k (\cos \Theta) \geq 0$ for different $k$ values !!
Monopole interaction is driver of changes in shell and sub-shell structure!!

Quadrupole interaction is key to the configuration mixing and collectivity that drives the evolution of structure!!
Between $^{40}\text{Zr}$ and $^{50}\text{Sn}$ protons fill $1g_{9/2}$ orbit. Large spatial overlap with neutron $1g_{7/2}$ orbit.

$1g_{7/2}$ orbit more tightly bound

Lower energy
INTRUDER STATE MODEL

PROTON LEVELS

NORMAL
2 VALENCE
NUCLEONS

INTRUDER
"6" VALENCE
NUCLEONS

NORMAL
6 VALENCE
NUCLEONS

Cd

Ba
Concept of monopole interaction changing shell structure and inducing collectivity
Seeing structural evolution
Different perspectives can yield different insights

Onset of deformation
as a phase transition
mediated by a change in shell structure

Note change in curves
from concave to convex
Two mechanisms for changes in magic numbers and shell gaps

• Changes in the single particle potential – occurs primarily far off stability where the binding of the last nucleons is very weak and their wave functions extend to large distances, thereby modifying the potential itself.

• Changes in single particle energies induced by the residual interactions, especially the monopole component.
$P_k (\cos \Theta)$

If actual force is short range (in $\Theta$) a low $k$ doesn’t reproduce rapid behavior of force with $\Theta$.
Need high $k$ which oscillates rapidly and can describe fine details.

\( \delta \) forces will have important high $k$ components

Long range forces: high $k$ NOT important because will average out due to oscillations in $P_k$ for high $k$
Quadrupole Interactions

Frequently, in nuclear physics calcs, a quadrupole force is used, alone or added to other forces.

Now we see why such a force can be important:

a) \textit{k limitation} for low spin orbits eliminates many higher multipoles

b) "\textit{finite (= not short) range}" forces weaken higher multipoles

\[ \delta + \text{Quad}: \text{approximates short and long range forces} \]

Pairing + Quad (PPQ) similar – very popular

Higher multipoles only important for both \( \frac{l}{j} > d_{5/2} \)
Off-diagonal effects

• Critically important to structure and structural evolution.

• Mix wave functions of the shell model, leading to collective effects and deformation.

• Same basic ideas, ways of thinking apply. Think in terms of angles between the particle orbits in different configurations relative to the angles at which the residual interaction is important. If the interaction “bridges” the angular difference and the energies of the configurations are close, they will likely mix considerably in the perturbed wave function.
Residual Interactions and Wave Functions

Configuration Mixing - Off Diagonal

\[ g_{7/2} \quad d_{5/2} \]

\[ \Psi = \left| d_{5/2}^2, J \right> \]

Same force can \[ \rightarrow \]

\[ \Psi = \left| \alpha d_{5/2} + \beta g_{7/2} \right> \]

\[ \psi \rightarrow \sum \alpha_i \Psi_i \]

Also: key to deformation (see later)

\[ \psi = \left| d_{5/2} g_{7/2}, J \right> \]
Can we estimate which configurations $|j_1, j_2, J\rangle$ will be important?

Relation between angles between $j_1, j_2$ in the 2 configurations and the angular dependence of the interaction.

Think of the interaction as “bridging” an angle $\Delta \Theta$

Determine which matrix elements

$$\langle A | V | B \rangle$$

are big for configs A, B.

Examples

1) $\langle \pi d_{5/2} \nu d_{5/2} J=1 | V | \pi d_{5/2} \nu d_{3/2} J=1 \rangle \rightarrow$ large int. for $\delta$ fct.
   $\Theta = 152^\circ \quad \Theta = 156^\circ$

2) $\langle d_{5/2}^2 J=4 | V | d_{5/2} d_{3/2} J=4 \rangle \rightarrow$ $\delta$: small interaction
   $\Theta = 82^\circ \quad \Theta = 49^\circ \quad Q$: moderate int.

Large interactions for forces that are large at angles equal to $\Delta \Theta$ of the two configs.
Multipole Decomposition of Residual Interactions

- Multipole order vs. range of force ($\theta$)
- Limitations on allowed multipoles
- Interplay of probe and “probee”
- Importance of quadrupole force
- Nature of $\delta$ interaction in different configurations

How different configurations sample the force

Forces and angles

Finite range (radial) forces