Multipole Decomposition of Residual Interactions

We have seen that the relative energies of 2-particle systems affected by a residual interaction depend SOLELY on the angles between the two angular momentum vectors, not on the radial properties of the interaction (which just give the scale).

We learn a lot by expanding the angular part of the residual interaction,

$$H_{residual} = V(\theta,\phi)$$

in spherical harmonics or Legendre polynomials.

Multipole Decomposition of Residual Interactions

Can expand an interaction (Θ) in series using complete set of functions P_k ($\cos\Theta$)

$v(\overline{r}_{12})$ is 2-body interaction. Write:

$$v(\overline{r}_{12}) = \sum_k v_k P_k(\cos\Theta)$$

Now, we re-consider the energy shifts

$$\Delta E\ (j_1, j_2, J) = \sum_k F_k^k A_k$$

where $F_k^r = \int R_{nl_1}^2\ R_{nl_2}^2\ v_k\ dr_1\ dr_2$

and $A = \dfrac{(-1)^{j_1+j_2+J}\ 4\pi}{2k+1} \left\langle l_1\, j_1 \| Y_k \| l_1\, j_1 \right\rangle$

$$\left\langle l_2\, j_2 \| Y_k \| l_2\, j_2 \right\rangle \propto \left\langle \begin{matrix} j_1 & j_2 & J \\ j_2 & j_1 & k \end{matrix} \right\rangle$$

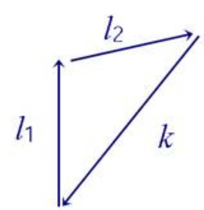
$\boxed{\text{Limitation on } k}$

$\boxed{\text{Key to physics}}$

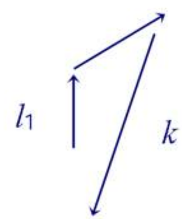
Relation of "probe" to force ! !

Triangle conditions: Y_k carries ang. mom. k

$\therefore$ l_1, l_2, k must "close"



ok



No

(Rigorously, applies to direct terms in interactions, not to exchange terms)

$$\boxed{\mathbf{k < min\ (2\mathit{l}_1,\ 2\mathit{l}_2,\ 2j_1,\ 2j_2)}}$$

k Restriction on Multipoles

Reflects important physical idea

Effects – great simplification

e.g.,	$d_{3/2}^2$	$k_{max} = 3$
	since k = even $\rightarrow$	$k = 0, 2$ only
	$d_{3/2}\ d_{5/2}$	$k = 0, 2$ only
	$d_{3/2}\ g_{9/2}$	$k = 0, 2$ only
	$d_{3/2}\ s_{1/2}$	$k = 0$ only !
	$s_{1/2}^2$	$k = 0$ only !
	$g_{7/2}^2$	$k = 0, 2, 4, 6$

Physical idea

Relation of probe to "probe"

2 particles in $s_{1/2}$ orbit are insensitive to multipoles above monopole (*i.e.,* constant) force ! ! What means?

Why?

Configuration for which only a monopole and/or quadrupole force applies

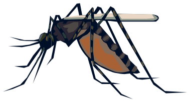
Monopole only	Monopole + Quadrupole	k = 0 + 2 + 4
	$d_{3/2}^2$	$d_{5/2}^2$
$s_{1/2}^2$, $p_{1/2}^2$	$p_{3/2}^2$	$f_{5/2}^2$
$s_{1/2}$ + anything	$d_{3/2}$ + anything	$d_{5/2}$ + anything
	$d_{3/2}$ + anything	$f_{5/2}$ + anything

All forces identical !

All forces effectively equivalent to a quadrupole force.

$\delta \sim Q$!

Probes and “probees”



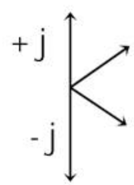
Probes and Objects Probed

Relative size of probe and object studied must be similar.

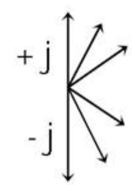
(Explains need for high energy accelerators for particle physics)

In our case, the probe (of the force) is the nucleon in an orbit, in particular the angular "size" (spread) of the wave function in a given magnetic substate, *m*, which is directly related to j.

low j



high j



s state: Ψ very spread out in space

$\therefore$ Can't sense details of a force. Particles "always" in contact. A δ force appears the same as a constant force ! !

High j – sensitive to details of force—each *m* value highly localized.

Multipole Decomposition of Residual Interactions

Can expand an interaction (Θ) in series using complete set of functions P_k ($\cos\Theta$)

ν ($\bar{r}_{12}$) is 2-body interaction. Write:

$$\nu\ (\bar{r}_{12}) = \sum_k \nu_k\ P_k\ (\cos\Theta)$$

Multiply on both sides by $P_{k'}$ ($\cos\Theta$) and integrate

$$\int \nu\left(\bar{r}_{12}\right) P_{k'}\left(\cos\Theta\right) d\,\Theta = \int \sum_k \nu_k\ P_k\left(\cos\Theta\right) P_{k'}\left(\cos\Theta\right) d\,\Theta$$

Use $Y_{k0}\ (\Theta) = \sqrt{\frac{2k+1}{4\pi}}\ P_k \cos\Theta$

$$\int \nu\left(\bar{r}_{12}\right) P_{k'}\left(\cos\Theta\right) d\,\Theta = \sqrt{\frac{4\pi}{2k+1}}\sqrt{\frac{4\pi}{2k'+1}}\int \nu_k\, Y_{k0}\left(\Theta\right) Y_{k'0}\left(\Theta\right) d\,\Theta$$

Use $\int Y_{km}\left(\Theta\right) d\,\Theta\, Y_{k'm'}\left(\Theta\right) \sin\Theta\, d\,\Theta\, d\,\Phi = \delta_{kk'}\ \delta_{mm'}$

$$\therefore \int \nu\left(\bar{r}_{12}\right) P_{k'}\left(\cos\Theta\right) d\,\Theta = \frac{4\pi}{2k+1}\ \frac{\nu_k}{2\pi}$$

$$\therefore \boxed{\nu_k = \frac{2k+1}{2}\int \nu\left(\bar{r}_{12}\right) P_k\left(\cos\Theta\right) d\,\Theta}$$

Expansion coefficient

Now, use this in the expression for the energy shifts in configuration $\left| j_1\, j_2\, J \right\rangle$ in presence of a residual interaction

Apply multipole expansion to δ force

For derivation, see Heyde

Result:

$$V_k(r_1, r_2) = \frac{2k+1}{4\pi} \frac{\delta(r_1 - r_2)}{r_1 \, r_2}$$

High k multipoles most important.

$k = 0$ component: monopole – overall shift of all J levels in $| \, j_1 \; j_2 \; J \, \rangle$

$k = 2$ same as quad. force

$\left(d_{3/2}^2 \right)$ $k = 0, 2$ only

Often difficult to distinguish forces from data on low j orbits.

Effects of Different Multipoles

Look at structure of:

$\left|\Psi_{12}\left(\Theta_{12}\right)\right|^2$ and of P_k (cos Θ)

Each 2-particle configuration, $|\, j_1\, j_2\, J\,\rangle$, will have Ψ_{max} (Θ_{12}) at the semi-classical angle and will fall off rapidly away from this angle.

e.g.,

$h^2_{11/2}$	J = 0	$\Psi_{max} \sim 180°$
	J = 2	$\Psi_{max} \sim 155°$
	J = 4	$\Psi_{max} \sim 134°$
	J = 6	$\Psi_{max} \sim 112°$
	J = 8	$\Psi_{max} \sim 84°$

If P_k (cos Θ) is large at one of those angles, an attractive force will lower the energy of that level

Reason: energy lowering ~ $\int \Psi^k\, P_k\, \Psi\, \delta\, \Theta$

Inte<u>grand</u> is only large when both Ψ (Θ), P_k (Θ) are large.

$|\frac{11}{2}\ \frac{11}{2}\ 5>$

Semi-classical angles for the configurations $|\, j_i j_k J >$

11/2

$J^\pi = 2^+$	1/2	3/2	5/2	7/2	9/2	13/2
1/2		63.4	133.1			
3/2	63.4	101.5	124.6	151.4		
5/2	133.1	124.6	131.1	142.0	159.1	
7/2		151.4	142.0	144.0	150.9	
9/2			159.1	150.9	151.5	166.5
13/2					166.3	159.8

11/2

J = 2+

Θ ~ 155°

$J^\pi = 4^+$	1/2	3/2	5/2	7/2	9/2	13/2
1/2				59.4	129.7	
3/2			49.1	88.1	116.2	
5/2		49.1	81.8	101.0	117.3	155.2
7/2	59.4	88.1	101.0	111.4	121.3	143.4
9/2	129.7	116.2	117.3	121.3	126.6	140.4
13/2			155.2	143.4	140.4	142.6

J = 4+

Θ ~ 134°

$J^\pi = 6^+$	1/2	3/2	5/2	7/2	9/2	13/2
1/2						128.3
3/2					45.5	112.8
5/2				41.8	73.2	112.0
7/2			41.8	70.5	87.8	114.0
9/2		45.5	73.2	87.8	98.7	117.0
13/2	128.3	112.8	112.0	114.0	117.0	124.7

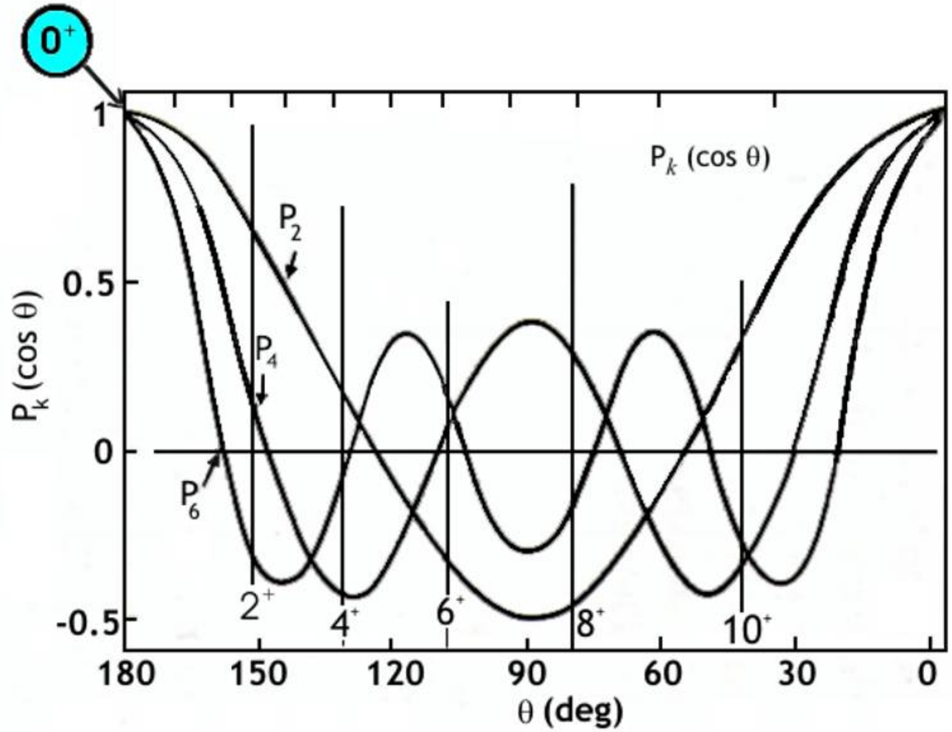
J = 6+

Θ ~ 112

$J^\pi = 8^+$	1/2	3/2	5/2	7/2	9/2	13/2
1/2						
3/2						43.9
5/2						69.4
7/2					37.1	82.2
9/2				37.1	63.0	91.2
13/2		43.9	69.4	82.2	91.2	105.2

J = 8+

Θ ~ 84°



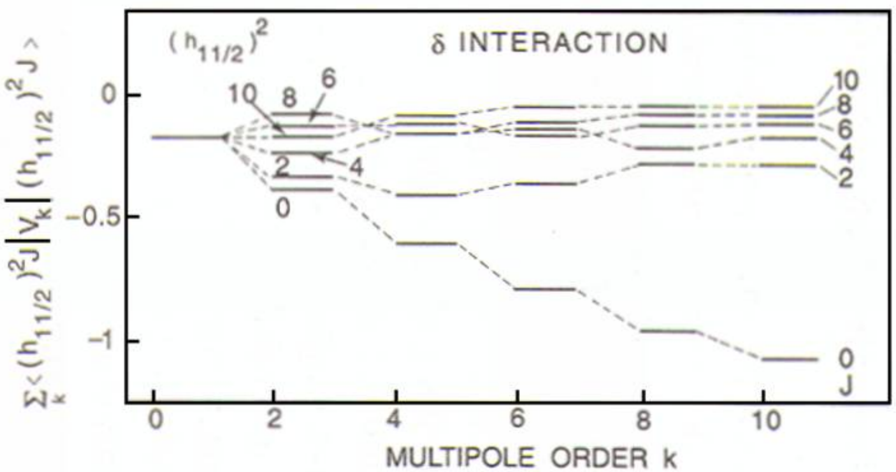
So — recall $V = -V_0\, \delta\, (\bar{r}_{12})$, so $P_k \rightarrow -P_k$: positive values are attractive

∴	k	2	4	6
	2^+	Attr.	Attr.	Rep.
	4^+	Attr.	Rep.	Rep.
	6^+	Rep.	~ 0	Attr.
	8^+	Rep.	Attr.	Rep.
	10^+	~0	Rep.	~0

Note: 0^+ always attractive – each k lowers
more k's allowed, more lowered
higher j

∴ 0^+ for higher j's (∴ higher k's allowed) lowered more

Multiplet Splittings for δ force in $(h_{11/2})^2$ configuration as function of multipoles included



$k = 0 + 2$	parabolic
All k	δ fct. results seen before

⇨ Separation of 0^+ from other states is specifically an effect of higher order terms—*i.e.*, shorter range terms. Pauli effect appears in highest orders!

⇨ Note: all multipoles affect 0^+ since $P_k(\cos 180^\circ) = 1$
For other spins cancellations occur.
$P_k(\cos \Theta) \gtrless 0$ for different k values !!

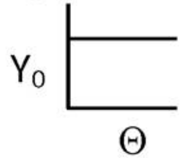
So, we are now in a position to consider

ANY FORCE, expressed through multipoles,

ANY CONFIGURATION, $|j_1 j_2\rangle$, for any J.

Consider special case: Monopole Force

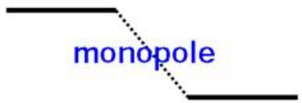
Y_0 (cos Θ) = constant



Different J's in $|j_1 j_2\, J\rangle$ differ only in Θ

So monopole force is equal for all Θ

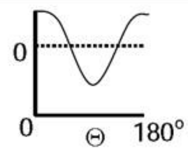
$|j_1 j_2\, J\rangle$



No splitting.

Effectively, $\boxed{\text{a shift to single particle energies.}}$
For $|j_2\, J\rangle$, called "self-energy". Leads to significant changes in s.p.e.'s as a function of N, Z ! !

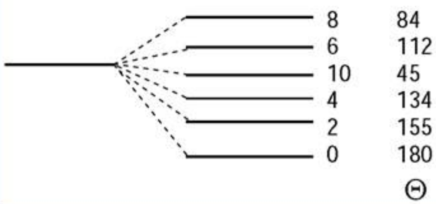
Quadrupole: Y_{20} (Θ) ~ $\cos^2 \Theta$ - 1

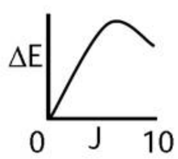


For a quadrupole force:

Δ E (J) is attractive (-) if Y_{20} (Θ) > 0 [Θ < 55°, > 125 °]
Δ E (J) is repulsive (+) if Y_{20} (Θ) < 0 [55° < Θ < 125 °]

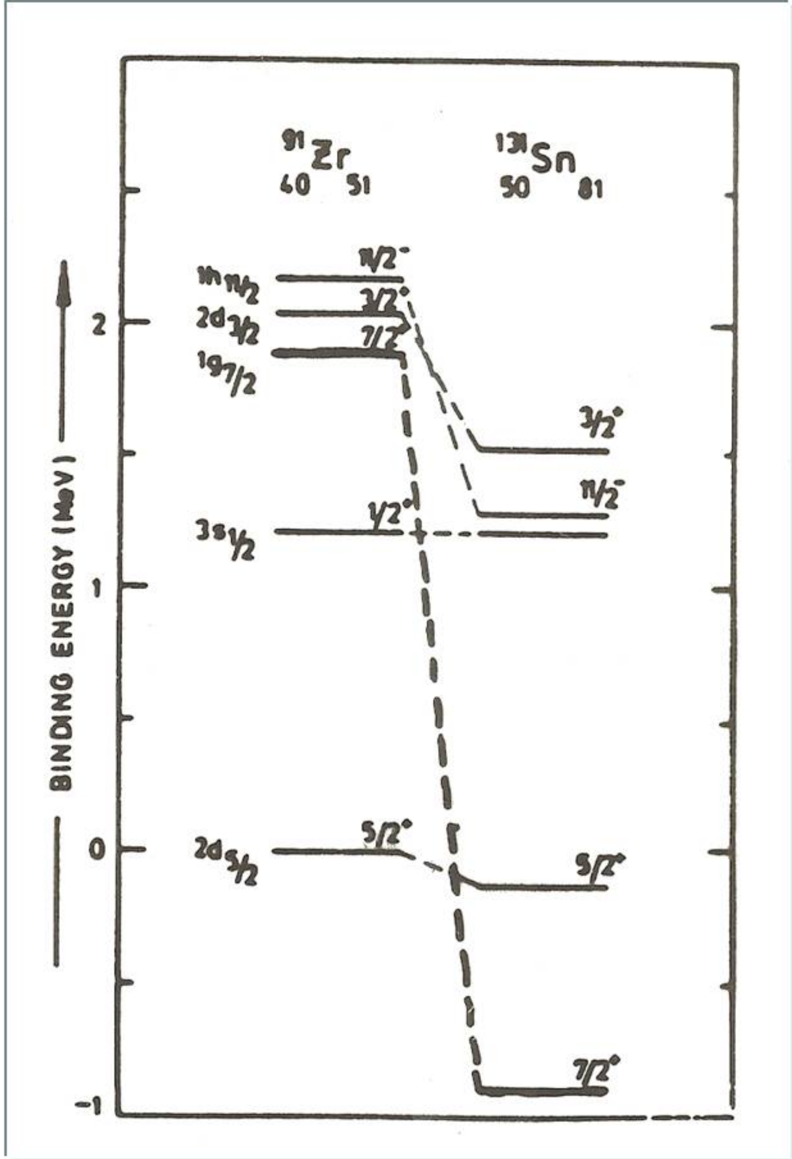
Example: $|h^2_{11/2}\, J\rangle$





Monopole interaction is driver of changes in shell and sub-shell structure !!

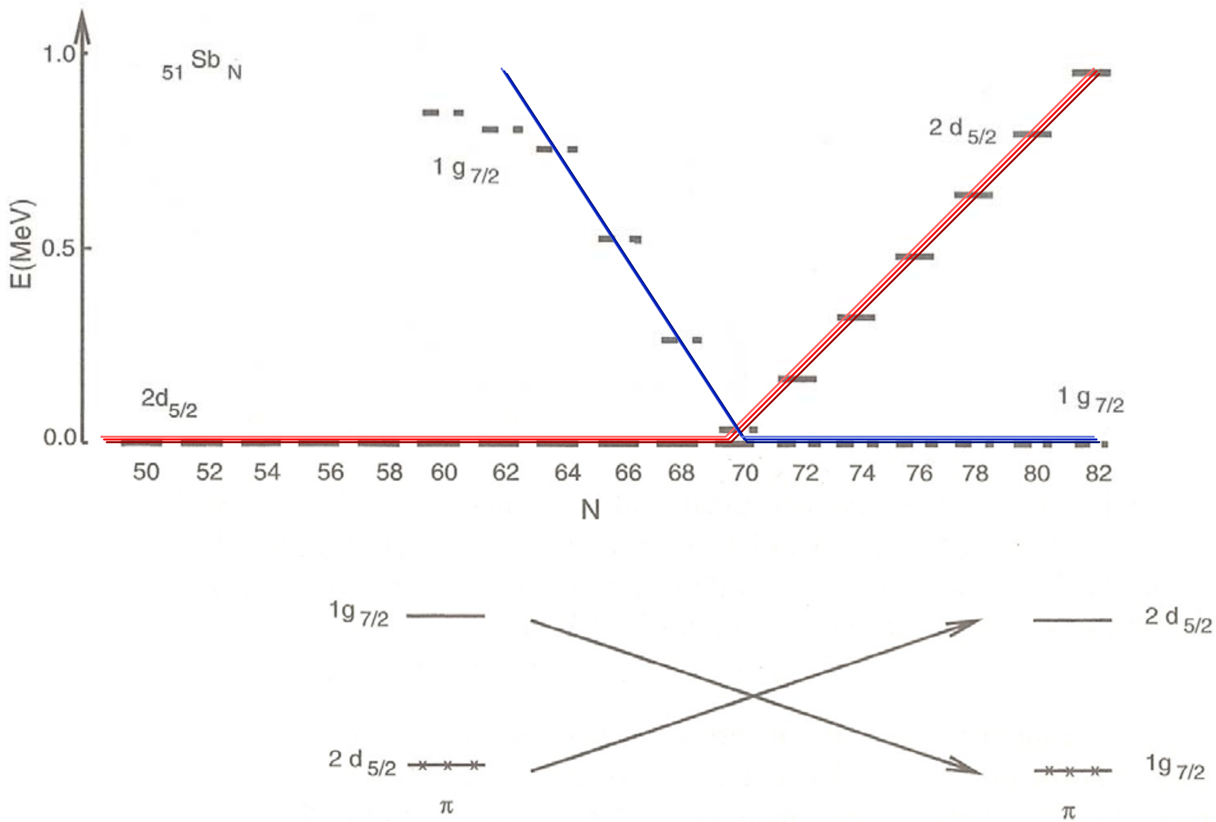
Quadrupole interaction is key to the configuration mixing and collectivity that drives the evolution of structure !!



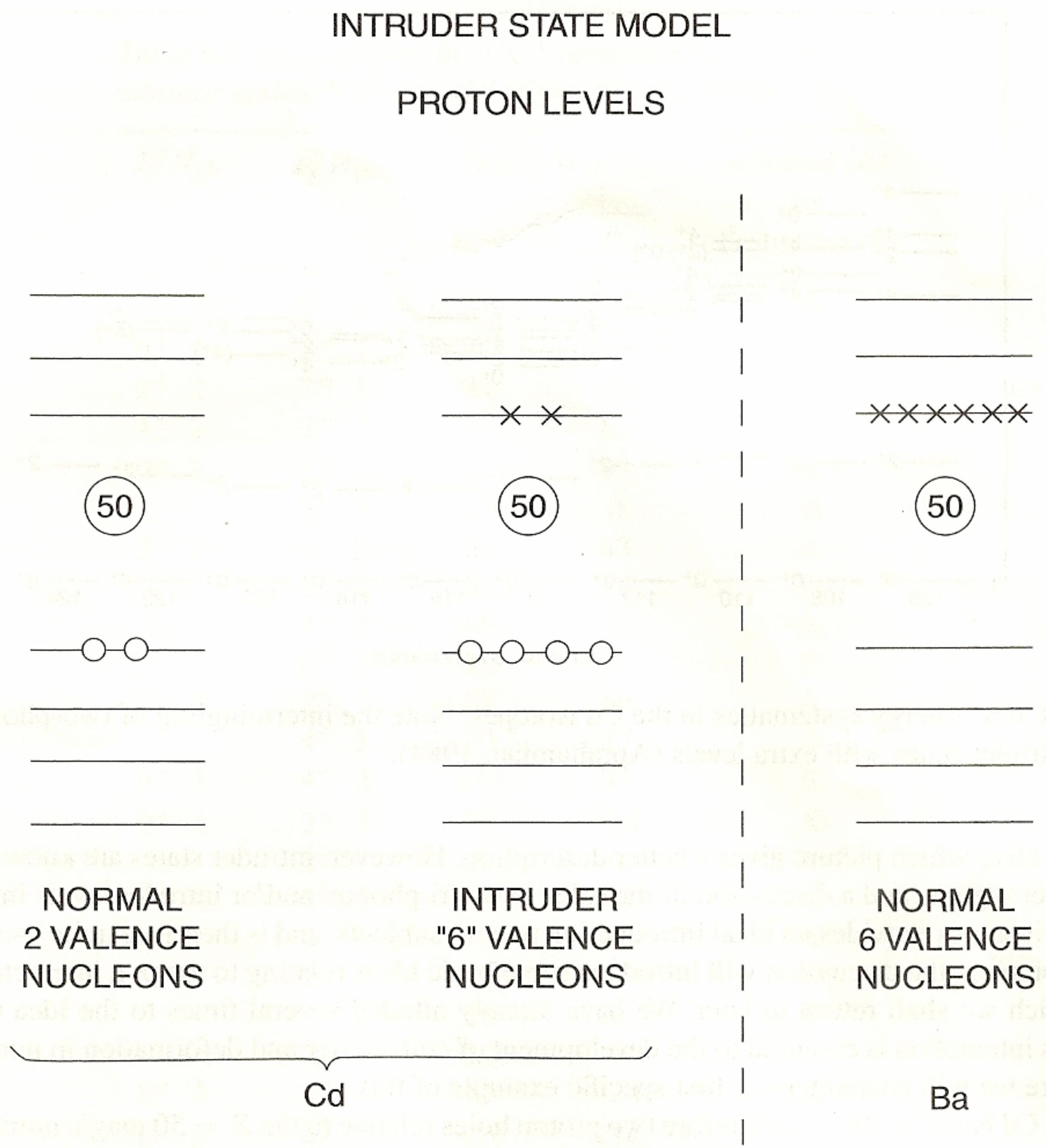
Between $_{40}$Zr and $_{50}$Sn protons fill 1$g_{9/2}$ orbit. Large spatial overlap with neutron 1$g_{7/2}$ orbit.

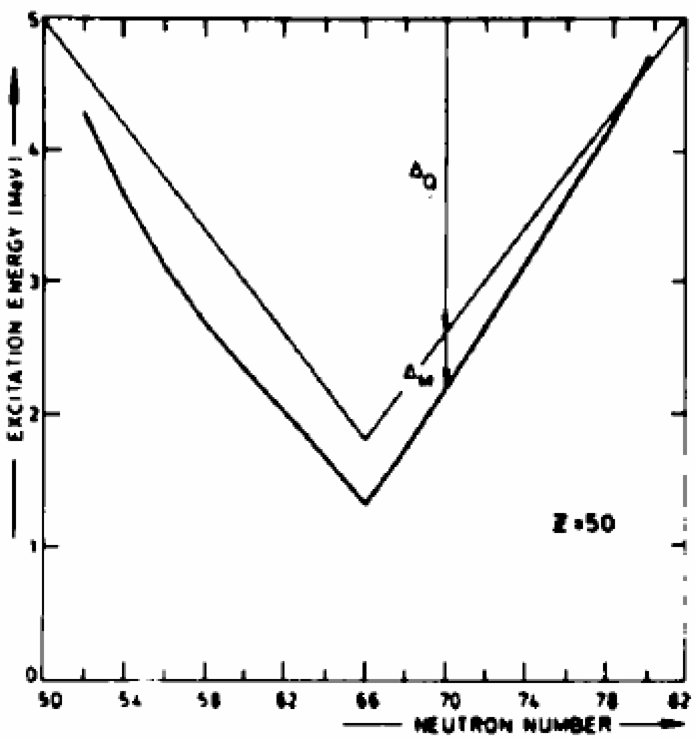
1$g_{7/2}$ orbit more tightly bound

Lower energy

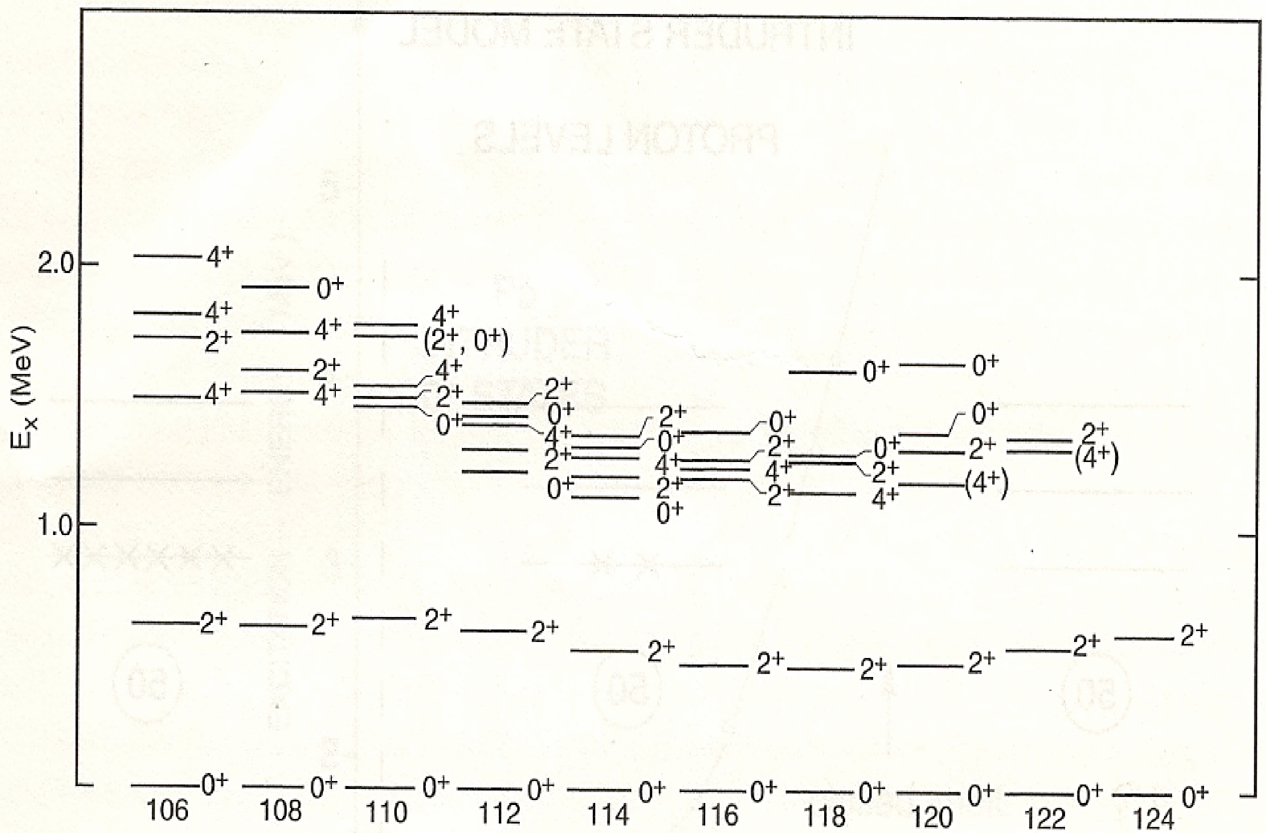


1.0
0.5
0.0
E(MeV)
$_{51}$Sb$_N$
1 g$_{7/2}$
2 d$_{5/2}$
2d$_{5/2}$
1 g$_{7/2}$
50 52 54 56 58 60 62 64 66 68 70 72 74 76 78 80 82
N
1g$_{7/2}$
2 d$_{5/2}$
π
2 d$_{5/2}$
1g$_{7/2}$
π

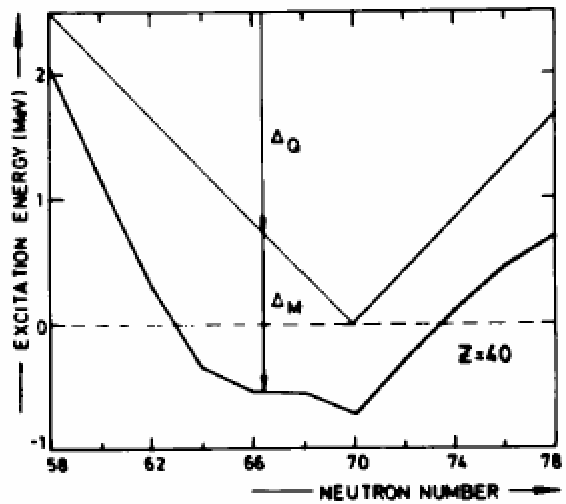

INTRUDER STATE MODEL
PROTON LEVELS
50
50
50
NORMAL
2 VALENCE
NUCLEONS
INTRUDER
"6" VALENCE
NUCLEONS
NORMAL
6 VALENCE
NUCLEONS
Cd
Ba

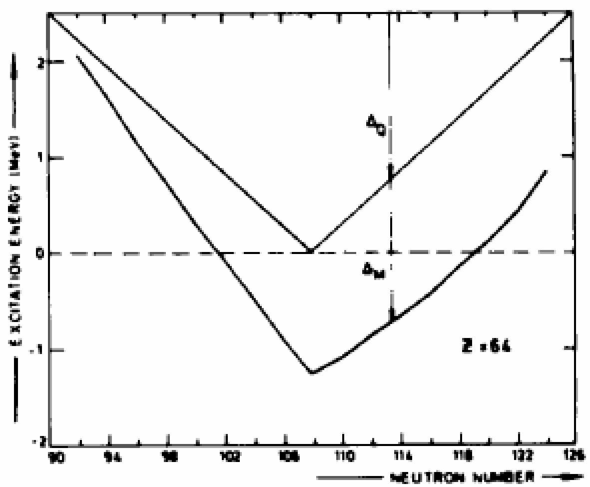


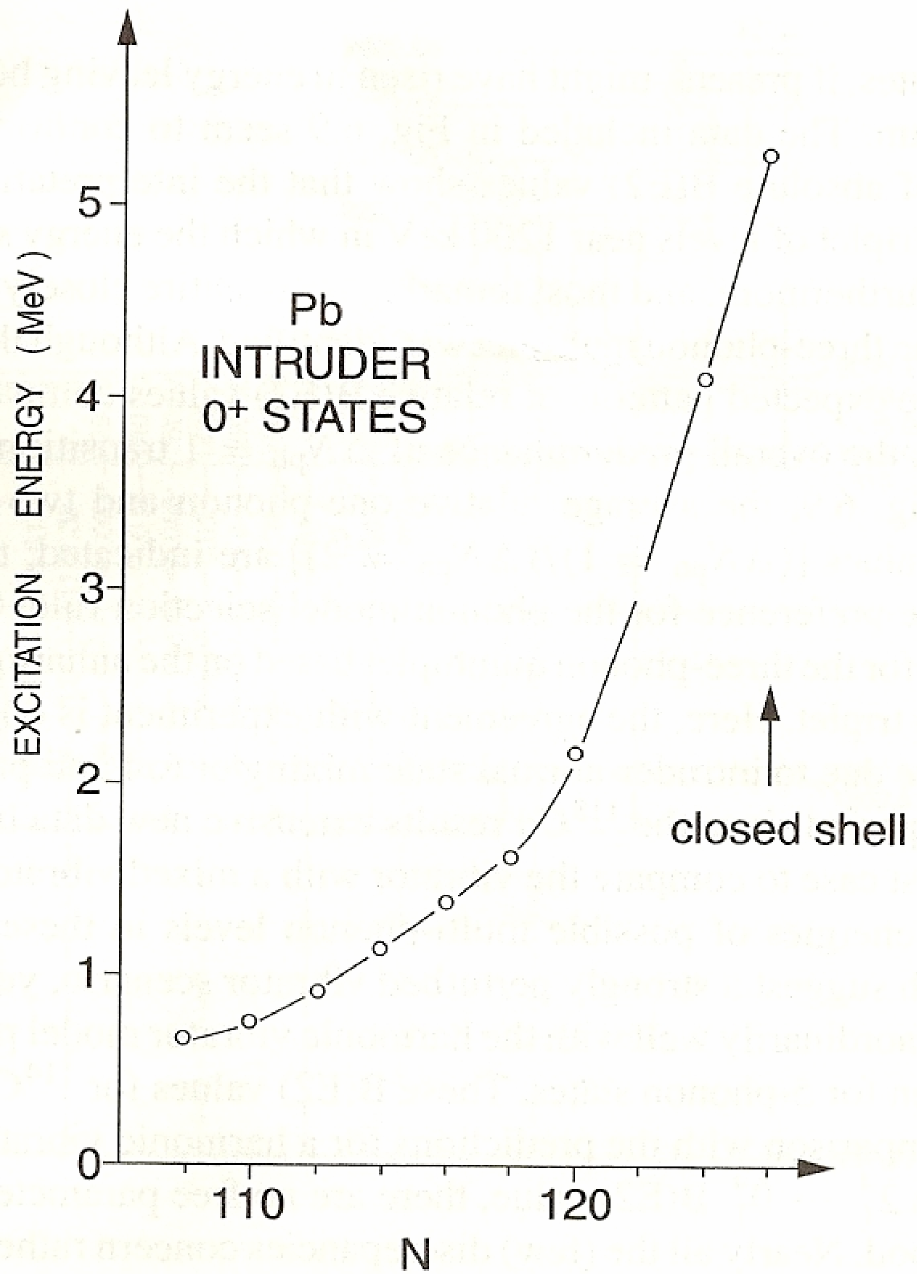
EXCITATION ENERGY (MeV)
NEUTRON NUMBER
Z = 50
Δ_Q
Δ_M
0
1
2
3
4
5
50
54
58
62
66
70
74
78
82



CADMIUM SYSTEMATICS

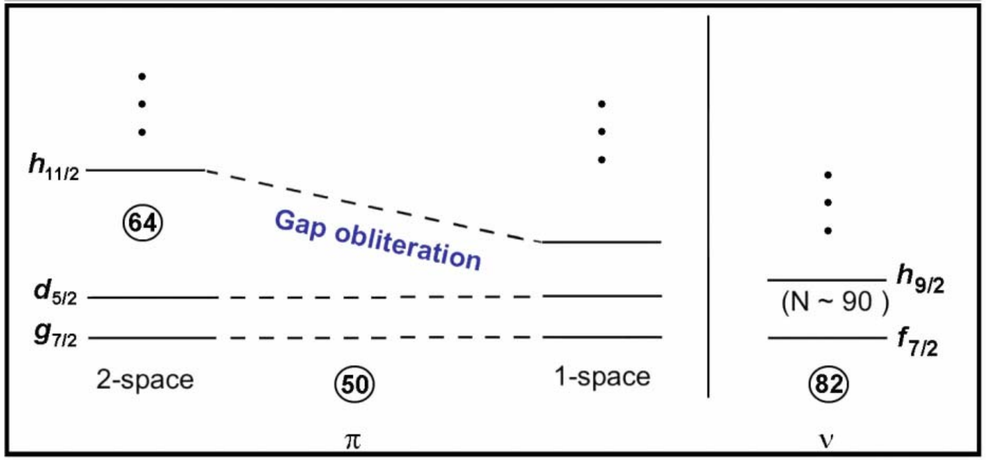

EXCITATION ENERGY (MeV)
2
1
0
-1
58
62
66
70
74
78
NEUTRON NUMBER
Δ_Q
Δ_M
Z=40

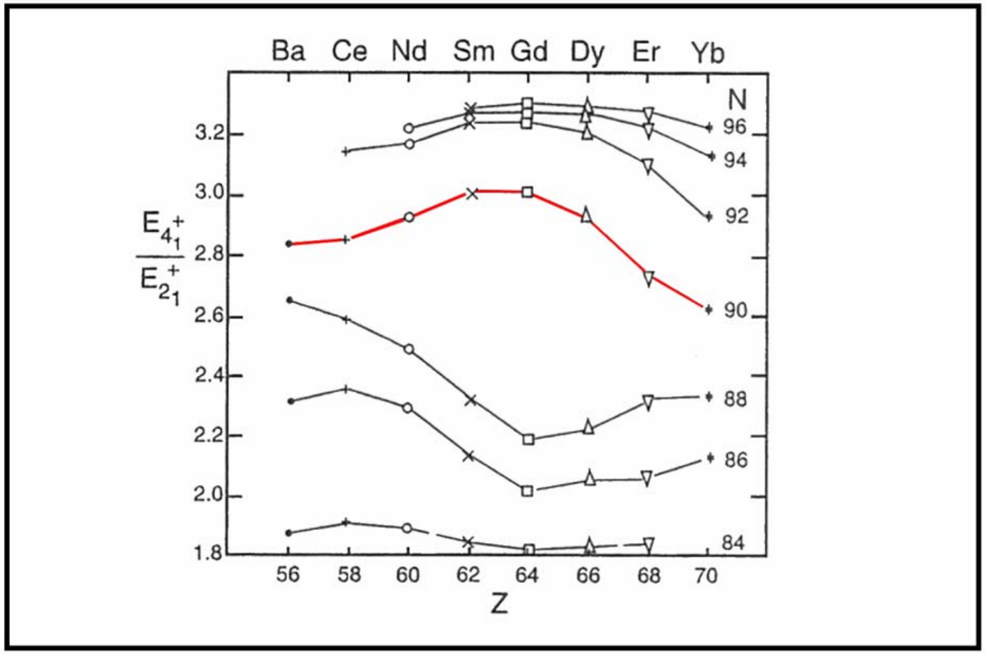

EXCITATION ENERGY (MeV)
2
1
0
-1
-2
90
94
98
102
106
110
114
118
122
126
NEUTRON NUMBER
Δ_Q
Δ_M
Z=64



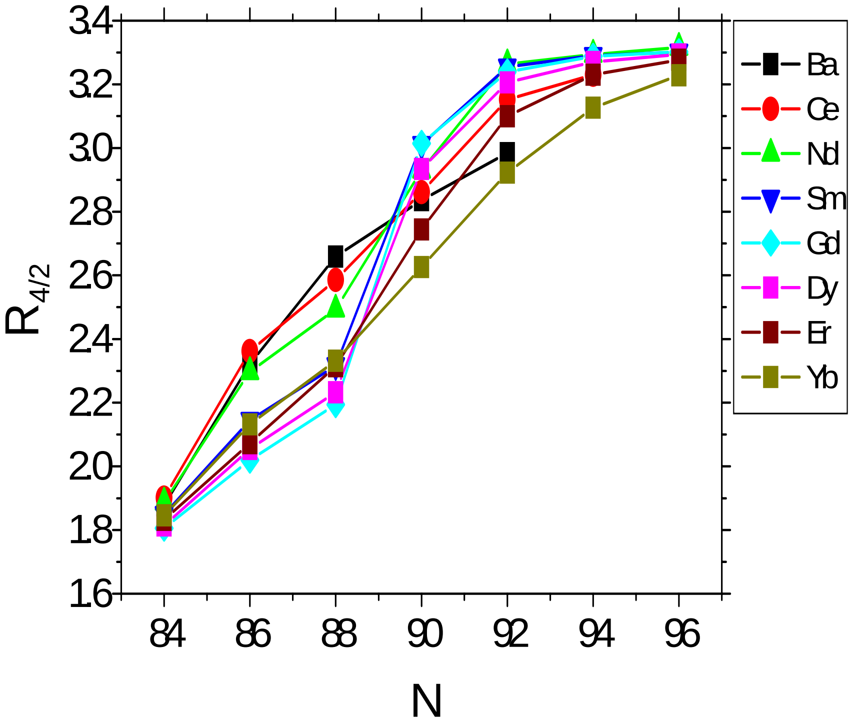
EXCITATION ENERGY (MeV)
Pb
INTRUDER
0+ STATES
closed shell
0
1
2
3
4
5
110
120
N

Concept of monopole interaction changing shell structure and inducing collectivity

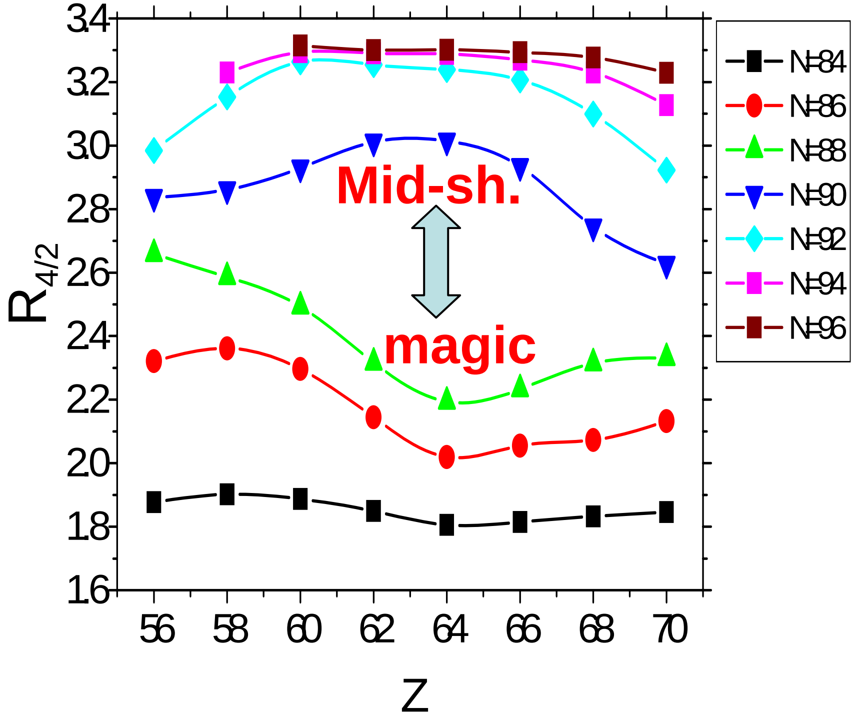




Seeing structural evolution
Different perspectives can yield different insights



Onset of deformation



Onset of deformation
as a phase transition
mediated by a change in shell structure

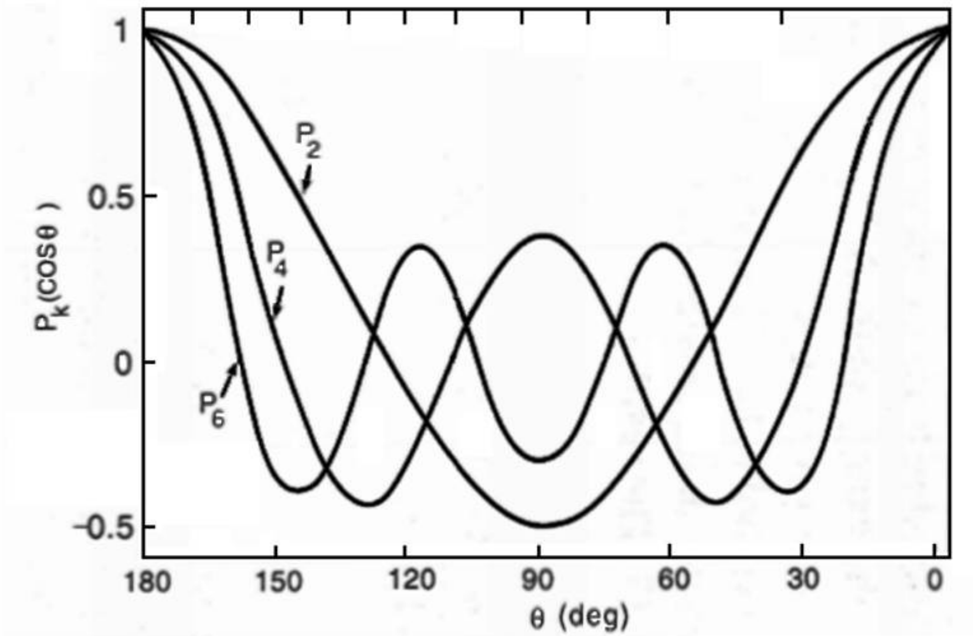
Note change in curves
from concave to convex

Two mechanisms for changes in magic numbers and shell gaps

- Changes in the single particle potential – occurs primarily far off stability where the binding of the last nucleons is very weak and their wave functions extend to large distances, thereby modifying the potential itself.

- Changes in single particle energies induced by the residual interactions, especially the monopole component.

P_k (cos Θ)



J = 0 ⟶ J_{max}

If actual force is short range (in Θ) a low *k* doesn't reproduce rapid behavior of force with Θ.
Need high *k* which oscillates rapidly and can describe fine details.

δ forces will have important high *k* components

Long range forces: high *k* NOT important because will average out due to oscillations in P_k for high *k*

!

Quadrupole Interactions

Frequently, in nuclear physics calcs, a quadrupole force is used, alone or added to other forces.

Now we see why such a force can be important:

a) *k* limitation for low spin orbits eliminates many higher multipoles

b) "finite (= not short) range" forces weaken higher multipoles

---

δ + Quad: approximates short and long range forces

Pairing + Quad (PPQ) similar – very popular

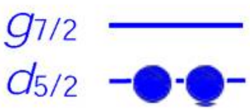
Higher multipoles only important for both "$l\,j$" > $d_{5/2}$

Off-diagonal effects

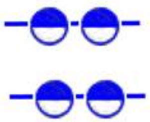
- Critically important to structure and structural evolution.
- Mix wave functions of the shell model, leading to collective effects and deformation.
- Same basic ideas, ways of thinking apply. Think in terms of angles between the particle orbits in different configurations relative to the angles at which the residual interaction is important. If the interaction “bridges” the angular diffference and the energies of the configurations are close, they will likely mix considerably in the perturbed wave function.

Residual Interactions and Wave Functions

Configuration Mixing – Off Diagonal



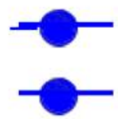
Same force can $\rightarrow$



$\Psi = | \, d^2_{5/2}, \, J \, \rangle$

$\Psi = \left| \alpha \, d_{5/2} + \beta \, g^2_{7/2} \; J \right\rangle$

or



$$\boxed{\Psi \rightarrow \sum_i \alpha_i \Psi_i}$$

Also: key to deformation (see later)

$\Psi = \left| d_{5/2} \, g_{7/2}, \, J \right\rangle$

Can we estimate which configurations $| j_1 j_2 J \rangle$ will be important

Relation between angles between $j_1\, j_2$ in the 2 configurations and the angular dependence of the interaction.

Think of the interaction as "bridging" an angle $\Delta\Theta$

Determine which matrix elements

$\langle A | V | B \rangle$ are big for configs A, B.

Examples

1) $\langle \pi d_{5/2}\, \nu d_{5/2}\, J=1 | V | \pi d_{5/2}\, \nu d_{3/2}\, J=1 \rangle \Longrightarrow$ large int. for δ fct.

$\Theta = 152^\circ \qquad \Theta = 156^\circ$

2) $\langle d^2_{5/2}\, J=4 | V | d_{5/2}\, d_{3/2}\, J=4 \rangle \Longrightarrow$ δ: small interaction

$\Theta = 82^\circ \quad \Theta = 49^\circ$ Q: moderate int.

Large interactions for forces that are large at angles equal to $\Delta\Theta$ of the two configs.

Multipole Decomposition

of

Residual Interactions

How different configurations sample the force

- Multipole order vs. range of force (θ)
- Limitations on allowed multipoles
- Interplay of probe and "probee"
- Importance of quadrupole force
- Nature of δ interaction in different configurations

Forces and angles

Finite range (radial) forces